Stochastic Processes

Week 04 (Version 1.3)

Poisson Processes

Point Process

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Outline of Week 04 Lectures

- Poisson Process
- Point Process

Recall: Binomial Distribution and its relation to Poisson Distribution

Bionomial Distribution: $X \sim B(n, p)$
probability of exactly k success in n trials:

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$B(n, p) \xrightarrow[np \text{ remains constant}]{n \to \infty} Poisson(np)$$

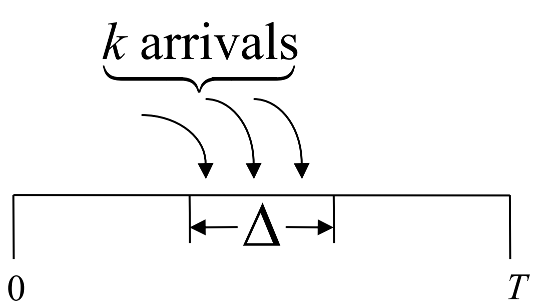
Poisson Processes

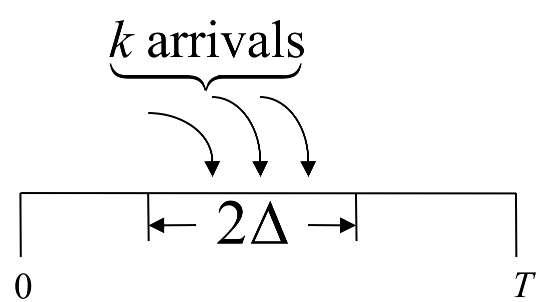
- Recall: Binomial and Poisson distributions:
  Both distributions can be used to model the number of occurrences of some event.
- Recall: Poisson arrivals are the limiting behavior of Binomial random variables. (Refer to Poisson approximation of Binomial random variables in your textbook):

$$P\left\{\begin{array}{l}\text{"}k \text{ arrivals occur in an} \\ \text{interval of duration } \Delta\text{"}\end{array}\right\} = e^{-\lambda}\frac{\lambda^k}{k!}, \qquad k = 0, 1, 2, \cdots$$

Where:

$$\lambda = np = \mu T \cdot \frac{\Delta}{T} = \mu\Delta$$





Poisson Processes

It follows that:

$$P\left\{\begin{array}{l}\text{"}k\text{ arrivals occur in an} \\ \text{interval of duration }2\Delta\text{"}\end{array}\right\} = e^{-2\lambda}\frac{(2\lambda)^k}{k!}, \qquad k = 0,\ 1,\ 2, \cdots,$$

since in that case:

$$np_1 = \mu T \cdot \frac{2\Delta}{T} = 2\mu\Delta = 2\lambda.$$

Poisson Processes

- Poisson arrivals over an interval form a Poisson random variable whose parameter depends on the duration of that interval.
- Moreover because of the Bernoulli nature of the underlying basic random arrivals, events over nonoverlapping intervals are independent.
- We shall use these two key observations to define a Poisson process formally.

Poisson Process

Definition: $X(t) = n(0, t)$ represents a Poisson process if:

(i) the number of arrivals $n(t_1, t_2)$ in an interval (t_1, t_2) of length $t = t_2 - t_1$ is a Poisson random variable with parameter λt. Thus:

$$P\{n(t_1, t_2) = k\} = e^{-\lambda t} \frac{(\lambda t)^k}{k!} \quad k = 0, 1, 2, \ldots, t = t_2 - t_1$$

And:

Poisson Processes

(ii) If the intervals (t_1, t_2) and (t_3, t_4) are nonoverlapping, then the random variables $n(t_1, t_2)$ and $n(t_3, t_4)$ are independent.

Since $n(0, t) \sim P(\lambda t)$ we have:

$$E[X(t)] = E[n(0,t)] = \lambda t$$

And:

$$E[X^2(t)] = E[n^2(0,t)] = \lambda t + \lambda^2 t^2$$

Poisson Processes

Autocorrelation function $R_{xx}(t_1, t_2)$:

$$R_{xx}(t_1, t_2) = E[X(t_1)X(t_2)]$$

$$X(t_1) = n(0, t_1) \; and \; X(t_2) = n(0, t_2)$$

To determine the autocorrelation function $R_{xx}(t_1, t_2)$ let $t_2 > t_1$ then from (ii) above $n(0, t_1)$ and $n(t_1, t_2)$ are independent Poisson random variables with parameters λt_1 and $\lambda(t_2 - t_1)$ respectively.

Thus:

$$E[n(0, t_1)n(t_1, t_2)] = E[n(0, t_1)]E[n(t_1, t_2)] = \lambda^2 t_1(t_2 - t_1)$$

But:

$$n(t_1, t_2) = n(0, t_2) - n(0, t_1) = X(t_2) - X(t_1)$$

And:

$$E[X(t_1)\{X(t_2) - X(t_1)\}] = R_{xx}(t_1, t_2) - E[X^2(t_1)]$$

We obtain:

$$R_{xx}(t_1, t_2) = \lambda^2 t_1(t_2 - t_1) + E[X^2(t_1)] = \lambda t_1 + \lambda^2 t_1 t_2$$
$$t_2 \geq t_1$$

Similarly, for $t_1 > t_2$:

$$R_{xx}(t_1, t_2) = \lambda t_2 + \lambda^2 t_1 t_2$$

Thus:

$$R_{xx}(t_1, t_2) = \lambda^2 t_1 t_2 + \lambda \min(t_1, t_2)$$

Poisson Distribution vs Poisson Processes

Poisson Distribution: A discrete probability distribution that expresses the probability of a given number of events occurring in a fixed interval of time or space.

Characteristics: It assumes that these events occur with a known constant mean rate and independently of the time since the last event.

Example: The number of emails received in an hour can be modeled using a Poisson distribution if emails arrive independently and at a constant average rate.

Poisson Distribution vs Poisson Processes

Poisson Process: A stochastic process that models a series of events occurring randomly over time or space.

Characteristics: It describes the occurrence of events that happen independently and at a constant average rate. The time between consecutive events follows an exponential distribution.

Example: The arrival of customers at a bank can be modeled as a Poisson process if the arrivals are independent and occur at a constant average rate.

Example:

$$X(t) \longrightarrow \boxed{\frac{d(\cdot)}{dt}} \longrightarrow X'(t)$$

(Derivative as a LTI system)

Then:

$$\mu_{X'}(t) = \frac{d\mu_X(t)}{dt} = \frac{d\lambda t}{dt} = \lambda, \quad a\ constant$$

And:

$$R_{XX'}(t_1, t_2) = \frac{\partial\, R_{XX}(t_1,\ t_2)}{\partial\ t_2} = \begin{cases} \lambda^2 t_1 & t_1 \le t_2 \\ \lambda^2 t_1 + \lambda & t_1 > t_2 \end{cases}$$

$$= \lambda^2 t_1 + \lambda\ U(t_1 - t_2)$$

And:

$$R_{X'X'}(t_1,\ t_2) = \frac{\partial\, R_{XX'}(t_1, t_2)}{\partial\, t_1} = \lambda^2 + \lambda\, \delta(t_1 -\ t_2).$$

Poisson Processes

Notice that:

- The Poisson process $X(t)$ *does not* represent a wide sense stationary process.

- Although $X(t)$ *does not* represent a wide sense stationary process, its derivative $X'(t)$ *does* represent a wide sense stationary process.

Poisson Processes

Since $X'(t)$ is a wide sense stationary process; nonstationary inputs to a LTI systems *can* lead to wide sense stationary outputs, an interesting observation!

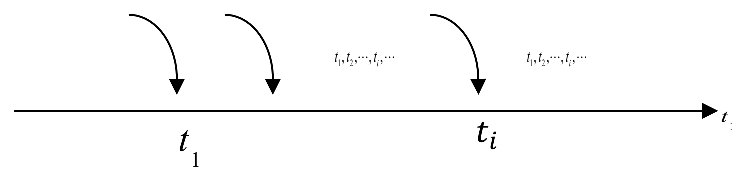
- **Sum of Poisson Processes:**

  If $X_1(t)$ and $X_2(t)$ represent two independent Poisson processes, then their sum $X_1(t) + X_2(t)$ is also a Poisson process with parameter $(\lambda_1 + \lambda_2)t$. (Follows from the definition of the Poisson process in (i) and (ii)).

Poisson Processes

Random selection of Poisson Points:

Let $t_1, t_2, \cdots, t_i, \cdots$ represent random arrival points associated with a Poisson process $X(t)$ with parameter λt, and associated with each arrival point, define an independent Bernoulli random variable N_i, where:



$$P(N_i = 1) = p, \qquad P(N_i = 0) = q = 1 - p.$$

Poisson Processes

Define the processes:

$$Y(t) = \sum_{i=1}^{X(t)} N_i \quad ; \quad Z(t) = \sum_{i=1}^{X(t)} (1 - N_i) = X(t) - Y(t)$$

We claim that both $Y(t)$ and $Z(t)$ are independent Poisson processes with parameters λpt and λqt, respectively, where $q = 1 - p$.
When X(t) is a Poisson process with parameter λt.

Poisson Processes

Proof:

$$Y(t) = \sum_{n=k}^{\infty} P\{Y(t)=k \mid X(t)=n\} P\{X(t)=n)\}.$$

But given $X(t) = n,$ we have $Y(t) = \sum_{i=1}^{n} N_i \;\sim\; B(n,p)$ so that:

$$P\{Y(t)=k \mid X(t)=n\} = \binom{n}{k} p^k q^{n-k}, \qquad 0 \le k \le n,$$

And:

$$P\{X(t)=n\} = e^{-\lambda t} \frac{(\lambda t)^n}{n!}.$$

$$P\{Y(t)=k\} = e^{-\lambda t}\sum_{n=k}^{\infty}\tfrac{n!}{(n-k)!k!}p^k q^{n-k}\tfrac{(\lambda t)^n}{n!} = \frac{p^k e^{-\lambda t}}{k!}\ (\lambda t)^k \underbrace{\sum_{n=k}^{\infty}\tfrac{(q\lambda t)^{n-k}}{(n-k)!}}_{e^{q\lambda t}}$$

$$= (\lambda pt)^k\ \frac{e^{-(1-q)\lambda t}}{k!} = e^{-\lambda pt}\ \frac{(\lambda pt)^k}{k!}, \qquad k=0,\,1,\,2,\,\cdots$$

$$\sim \quad P(\lambda pt).$$

More generally:

$$P\{Y(t)=k, Z(t)=m\} = P\{Y(t)=k, X(t)-Y(t)=m\}$$

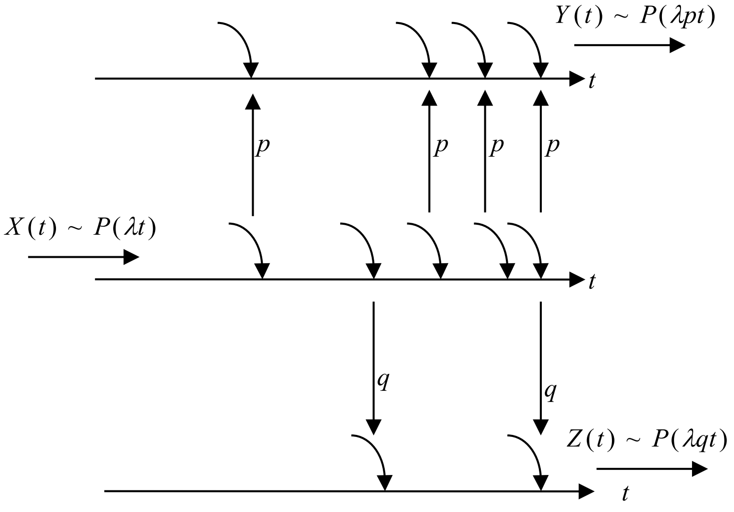
$$= P\{Y(t)=k, X(t)=k+m\}$$

$$= P\{Y(t)=k \mid X(t)=k+m\}P\{X(t)=k+m\}$$

$$= \binom{k+m}{k}p^k q^m \cdot e^{-\lambda t}\ \frac{(\lambda t)^{k+m}}{(k+m)!} = \underbrace{e^{-\lambda pt}\ \frac{(\lambda pt)^n}{k!}}_{P(Y(t)=k)}\quad \underbrace{e^{-\lambda qt}\ \frac{(\lambda qt)^n}{m!}}_{P(Z(t)=m)}$$

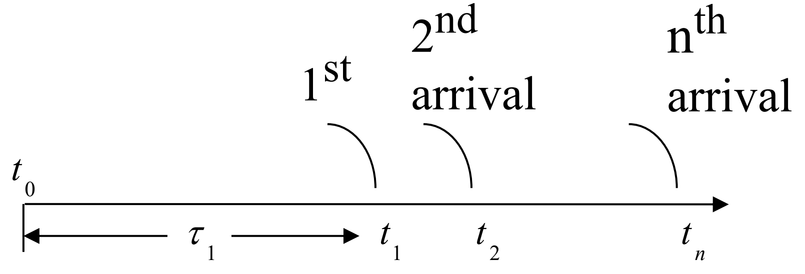
$$= P\{Y(t)=k\}P\{Z(t)=m\},$$

- Notice that $Y(t)$ and $Z(t)$ are generated as a result of random Bernoulli selections from the original Poisson process $X(t)$, where each arrival gets tossed over to either $Y(t)$ with probability p or to $Z(t)$ with probability q. Each such sub-arrival stream is also a Poisson process. Thus, a random selection of Poisson points preserves the Poisson nature of the resulting processes.
- However, a deterministic selection from a Poisson process destroys the Poisson property for the resulting processes.



Inter-arrival Distribution for Poisson Processes

Let τ_1 denote the time interval (delay) to the first arrival from *any* fixed point t_0. To determine the probability distribution of the random variable τ_1, we argue as follows: Observe that the event "$\tau_1 > t$" is the same as "$n(t_0, t_0+t) = 0$", or the complement event "$\tau_1 \le t$" is the same as the event "$n(t_0, t_0+t) > 0$" .



Inter-arrival Distribution for Poisson Processes

Hence the distribution function of τ_1 is given by:

$$F_{\tau_1}(t) \triangleq P\{\tau_1 \le t\} = P\{X(t) > 0\} = P\{n(t_0, t_0 + t) > 0\}$$
$$= 1 - P\{n(t_0, t_0 + t) = 0\} = 1 - e^{-\lambda t}$$

Its derivative gives the probability density function for τ_1 to be:

$$f_{\tau_1}(t) = \frac{dF_{\tau_1}(t)}{dt} = \lambda e^{-\lambda t}, \qquad t \ge 0$$

i.e. τ_1 is an exponential random variable with parameter λ so that: $E(\tau_1) = 1/\lambda.$

Inter-arrival Distribution for Poisson Processes

Similarly, let t_n represent the n^{th} random arrival point for a Poisson process. Then:

$$\overset{\Delta}{} F_{t_n}(t) = P\{t_n \le t\} = P\{X(t) \ge n\}$$

$$= 1 - P\{X(t) < n\} = 1 - \sum_{k=0}^{n-1} \frac{(\lambda t)^k}{k!} e^{-\lambda t}$$

and hence:

$$f_{t_n}(x) = \frac{dF_{t_n}(x)}{dx} = -\sum_{k=1}^{n-1} \frac{\lambda (\lambda x)^{k-1}}{(k-1)!} e^{-\lambda x} + \sum_{k=0}^{n-1} \frac{\lambda (\lambda x)^{k}}{k!} e^{-\lambda x}$$

$$= \frac{\lambda^n x^{n-1}}{(n-1)!} e^{-\lambda x}, \qquad x \ge 0$$

Inter-arrival Distribution for Poisson Processes

which represents a Gamma density function. i.e., the waiting time to the n^{th} Poisson arrival has a Gamma distribution.
Moreover:

$$t_n = \sum_{i=1}^{n} \tau_i$$

where τ_i is the random inter-arrival duration between the $(i-1)^{th}$ and i^{th} events. Notice that τ_i s are independent, identically distributed random variables. Hence using their characteristic functions, it follows that all inter-arrival durations of a Poisson process are independent exponential random variables with common parameter λ.
i.e.,

$$f_{\tau_i}(t) = \lambda e^{-\lambda t}, \qquad t \geq 0.$$

Inter-arrival Distribution for Poisson Processes

Alternatively, we have τ_1 is an exponential random variable.
By repeating that argument after shifting t_0 to the new point t_1, we conclude that τ_2 is an exponential random variable.
Thus, the sequence $\tau_1, \tau_2, \cdots, \tau_n, \cdots$ are independent exponential random variables with common p.d.f.

Thus, if we systematically tag every m^{th} outcome of a Poisson process $X(t)$ with parameter λt to generate a new process $e(t)$, then the inter-arrival time between any two events of $e(t)$ is a Gamma random variable.

Inter-arrival Distribution for Poisson Processes

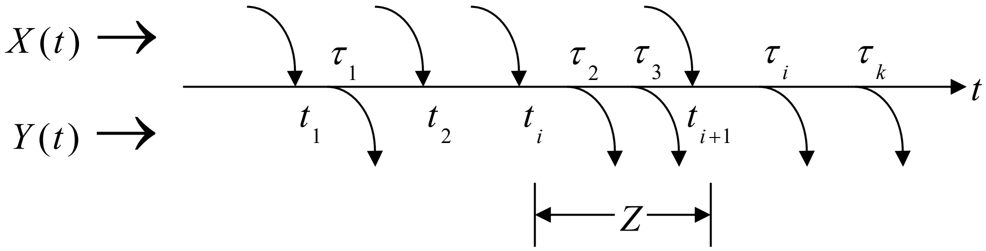
Notice that:

$$E[e(t)] = m/\lambda, \text{ and if } \lambda = m\mu, \text{ then } E[e(t)] = 1/\mu.$$

The inter-arrival time of $e(t)$ in that case represents an Erlang-m random variable, and $e(t)$ is an Erlang-m process.
In summary, if Poisson arrivals are randomly redirected to form new queues, then each such queue generates a new Poisson process.

Poisson Departures between Exponential Inter-arrivals

Let $X(t) \sim P(\lambda t)$ and $Y(t) \sim P(\mu t)$ represent two independent Poisson processes called *arrival* and *departure* processes.



Let Z represent the random interval between *any* two successive arrivals of $X(t)$. Z has an exponential distribution with parameter λ. Let N represent the number of "departures" of $Y(t)$ between *any* two successive arrivals of $X(t)$. Then from the Poisson nature of the departures we have:

$$P\{N = k \mid Z = t\} = e^{-\mu t}\frac{(\mu t)^k}{k!}.$$

Poisson Departures between Exponential Inter-arrivals

$$
\begin{aligned}
P\{N=k\} &= \int_0^\infty P\{N=k \mid Z=t\} f_Z(t)\,dt \\
&= \int_0^\infty e^{-\mu t} \tfrac{(\mu t)^k}{k!} \lambda e^{-\lambda t}\,dt \\
&= \frac{\lambda}{k!} \int_0^\infty (\mu t)^k e^{-(\lambda+\mu)t}\,dt \\
&= \frac{\lambda}{\lambda+\mu} \left(\frac{\mu}{\lambda+\mu}\right)^k \frac{1}{k!} \underbrace{\int_0^\infty x^k e^{-x}\,dx}_{k!} \\
&= \left(\frac{\lambda}{\lambda+\mu}\right) \left(\frac{\mu}{\lambda+\mu}\right)^k, \qquad k = 0,\ 1,\ 2,\ \cdots
\end{aligned}
$$

.

Poisson Departures between Exponential Inter-arrivals

- The random variable N has a geometric distribution. Thus if customers come in and get out according to two independent Poisson processes at a counter, then the number of arrivals between any two departures has a geometric distribution.

- Similarly, the number of departures between *any* two arrivals also represents another geometric distribution.

Example

Suppose there are 2 independent Poisson processes with $\lambda_1 = 1, \lambda_2 = 2$.
Find the probability that 2nd arrival of first process occurs before 3rd arrival of the second process.
Solution:
Consider the superposition of these two Poisson processes. It is still a Poisson process with $\lambda = 1 + 2 = 3$. Also, each event of the resulting process is from first process with probability $\frac{\lambda_1}{\lambda_1+\lambda_2} = \frac{1}{3}$ and otherwise with probability $\frac{2}{3}$. So, for the 2nd arrival of first process to occur before 3rd arrival of the second process. The number of arrivals from the first process in the first k arrivals of the combined process follows a binomial distribution. what is the probability that in the first 2+3-1=4 arrivals of the combined process, at most 2-1=1 of them are from the first process?
Let X be the number of arrivals from the first process in the first 4 combined arrivals. X is Binomial(n=4,p=1/3). find P(X<2)=P(X=0)+P(X=1):
we need the first 4 occurrences to cover at least 2 occurrences of the first process:

$$\sum_{k=2}^{4} \binom{4}{k} \left(\frac{1}{3}\right)^{k} \left(\frac{2}{3}\right)^{4-k}$$

P(X=0) = 16/81
P(X=1) = 32/81
P(X<2) = (16/81) + (32/81) = 48/81 = 16/27

Example: Coupon Collecting

Suppose a cereal manufacturer randomly inserts a sample of one type of coupon into each cereal box. Suppose there are n such distinct types of coupons. One interesting question is how many boxes of cereal should one buy on average to collect at least one coupon of each kind?

Example: Coupon Collecting

We shall reformulate the above problem in terms of Poisson processes. Let $X_1(t), X_2(t), \cdots, X_n(t)$ represent *n independent* identically distributed Poisson processes with common parameter λt. Let $t_{i1}, t_{i2}, \cdots$ represent the first, second, … random arrival instants of the process $X_i(t), \quad i = 1, 2, \cdots, n.$ They will correspond to the first, second, $\cdots$ appearance of the i^{th} type of coupon in the above problem. Let:

$$X(t) \triangleq \sum_{i=1}^{n} X_i(t),$$

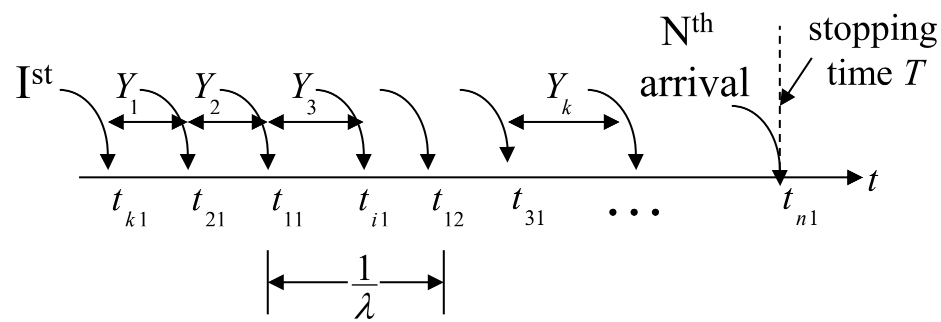
so that the sum $X(t)$ is also a Poisson process with parameter μt, where

$$\mu = n\lambda.$$

Example: Coupon Collecting

$1/\lambda$ represents: The average inter-arrival duration between any two arrivals of $X_i(t), i = 1, 2, \cdots, n,$ whereas:

$1/\mu$ represents the average inter-arrival time for the combined sum process *X*(*t*).



Example: Coupon Collecting

Stage 1: Getting the first coupon. You need to buy only one box to get a coupon you don't have. The probability of success is n/n=1. The expected number of boxes for this stage is 1.

Stage k: Getting the k-th distinct coupon. You have k-1 unique coupons. The probability of getting a new one is (n-(k-1))/n. The expected number of boxes for this stage is n/(n-(k-1)).

Stage n: Getting the final (n-th) distinct coupon. You have (n-1) unique coupons. The probability of getting the last one is (1/n). The expected number of boxes for this stage is n/1=n.

The total expected number of boxes is the sum of the expected values for each stage:

$$E[X] = 1 + \frac{n}{n-1} + \frac{n}{n-2} + \ldots + \frac{n}{1}$$

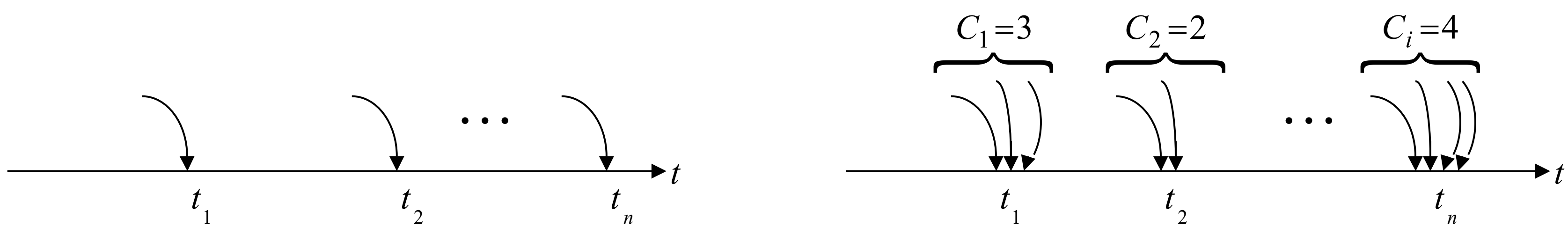
$$E[X] = n \times \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right)$$

Bulk Arrivals and Compound Poisson Processes

In an ordinary Poisson process $X(t)$, only one event occurs at any arrival instant. Instead suppose a random number of events C_i occur simultaneously as a cluster at every arrival instant of a Poisson process. If $X(t)$ represents the total number of all occurrences in the interval $(0, t)$, then $X(t)$ represents a **compound Poisson process**, or a **bulk arrival process**.

Bulk Arrivals and Compound Poisson Processes

Inventory orders, arrivals at an airport queue, tickets purchased for a show, etc. follow this process (when things happen, they happen in a bulk, or a bunch of items are involved.)



(a) Poisson Process

(b) Compound Poisson Process

Let:

$$p_k = P\{C_i = k\}, \quad k = 0, 1, 2, \cdots$$

represent the common probability mass function for the occurrence in any cluster C_i. Then the compound process $X(t)$ satisfies:

$$X(t) = \sum_{i=1}^{N(t)} C_i ,$$

where $N(t)$ represents an ordinary Poisson process with parameter λ. Let:

$$P(z) = E\{z^{C_i}\} = \sum_{k=0}^{\infty} p_k z^k$$

represent the moment generating function associated with the cluster Statistics. Then the moment generating function of the compound Poisson process $X(t)$ is given by:

$$\phi_X(z) = \sum_{n=0}^{\infty} z^n P\{X(t) = n\} = E\{z^{X(t)}\}$$

$$= E\{E[z^{X(t)} \mid N(t) = k]\} = E[E\{z^{\sum_{i=1}^{k} C_i} \mid N(t) = k\}]$$

$$= \sum_{k=0}^{\infty} (E\{z^{C_i}\})^k P\{N(t) = k\}$$

$$= \sum_{k=0}^{\infty} P^k(z) e^{-\lambda t} \frac{(\lambda t)^k}{k!} = e^{-\lambda t(1-P(z))}$$

If we let:

$$P^k(z) \triangleq \left(\sum_{n=0}^{\infty} p_n z^k \right)^k = \sum_{n=0}^{\infty} p_n^{(k)} z^n$$

where $\{p_n^{(k)}\}$ represents the k fold convolution of the sequence $\{p_n\}$ with itself, we obtain:

$$P\{X(t)=n\} = \sum_{k=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^k}{k!} p_n^{(k)}$$

The above, represents the probability that there are n arrivals in the interval $(0, t)$ for a compound Poisson process $X(t)$.

We can rewrite $\phi_X(z)$ also as:

$$\phi_X(z) = e^{-\lambda_1 t(1-z)} e^{-\lambda_2 t(1-z^2)} \cdots e^{-\lambda_k t(1-z^k)} \cdots$$

where $\lambda_k = p_k \lambda,$ which shows that the compound Poisson process can be expressed as the sum of integer-scaled independent Poisson processes $m_1(t), m_2(t), \cdots$. Thus:

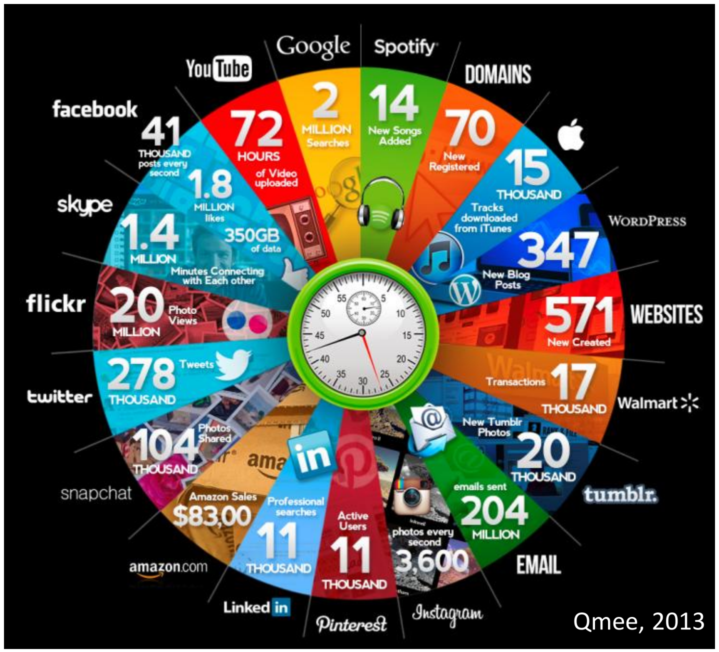
$$X(t) = \sum_{k=1}^{\infty} k\, m_k(t).$$

More generally, every linear combination of independent Poisson processes represents a compound Poisson process.

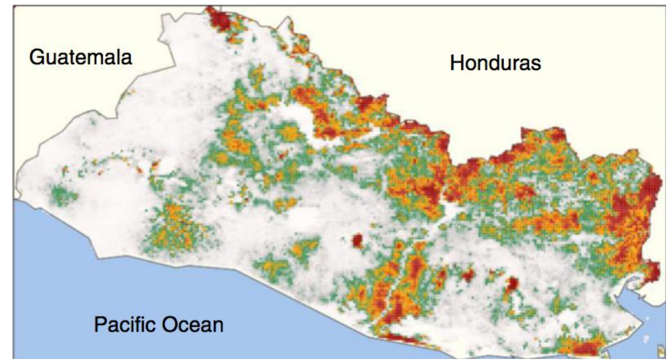
Outline of Week 04 Lectures

- Poisson Process
- Point Process

Many discrete *events* in continuous time

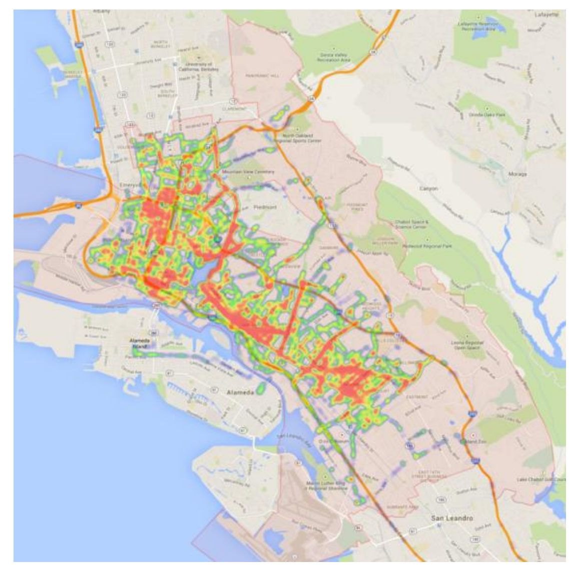


Online actions



Disease dynamics

Financial trading



Mobility dynamics

Variety of processes behind these events

Events are (noisy) observations of a variety of complex dynamic processes…

Stock trading

Flu spreading

Article creation in Wikipedia

News spread in Twitter

Reviews and sales in Amazon

Ride-sharing requests

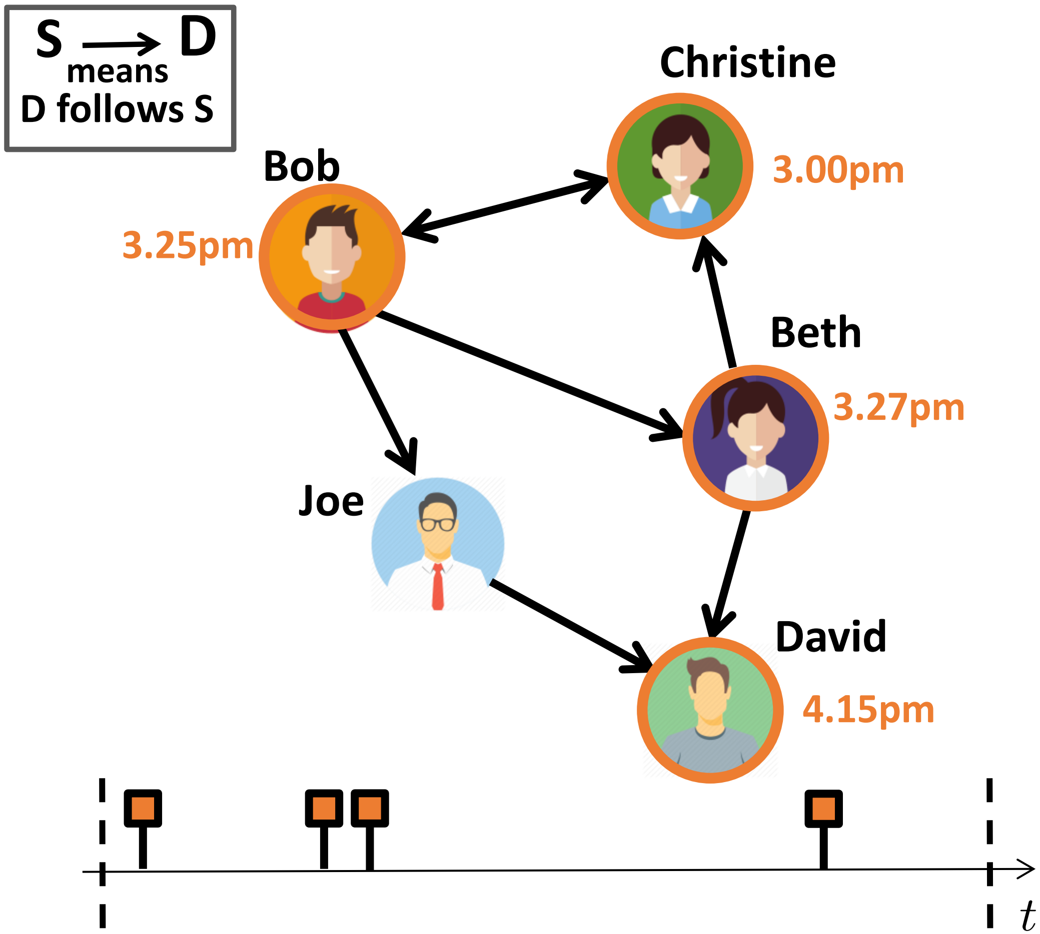
A user's reputation in Quora

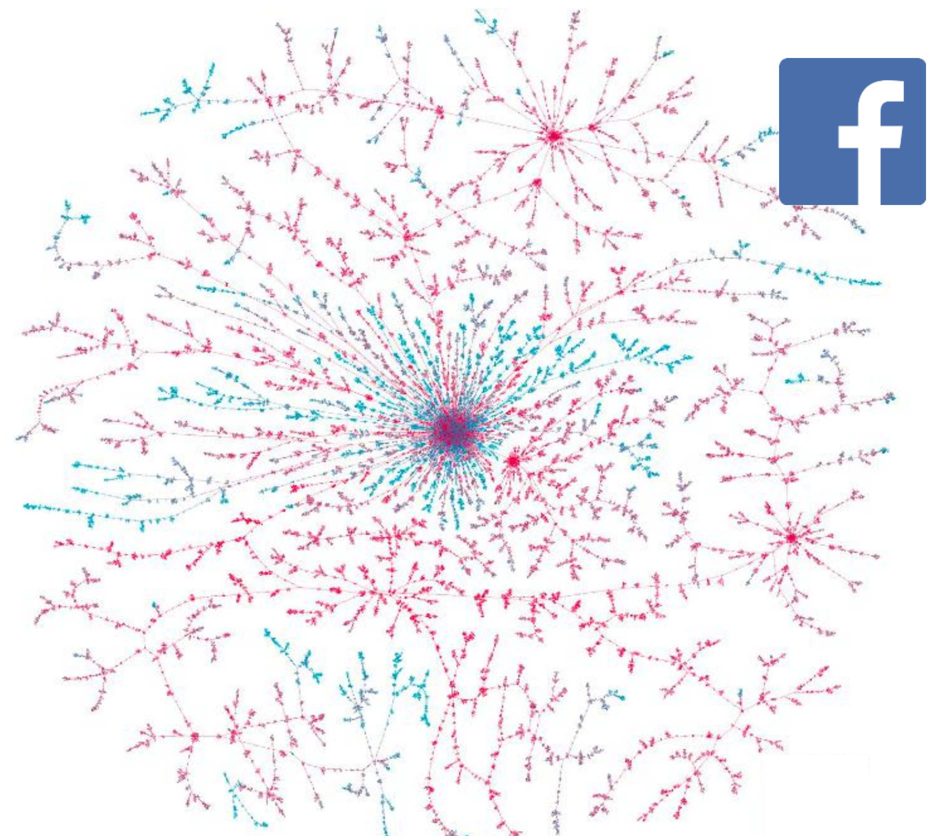
FAST

SLOW

…in a wide range of temporal scales.

Example I: Information propagation



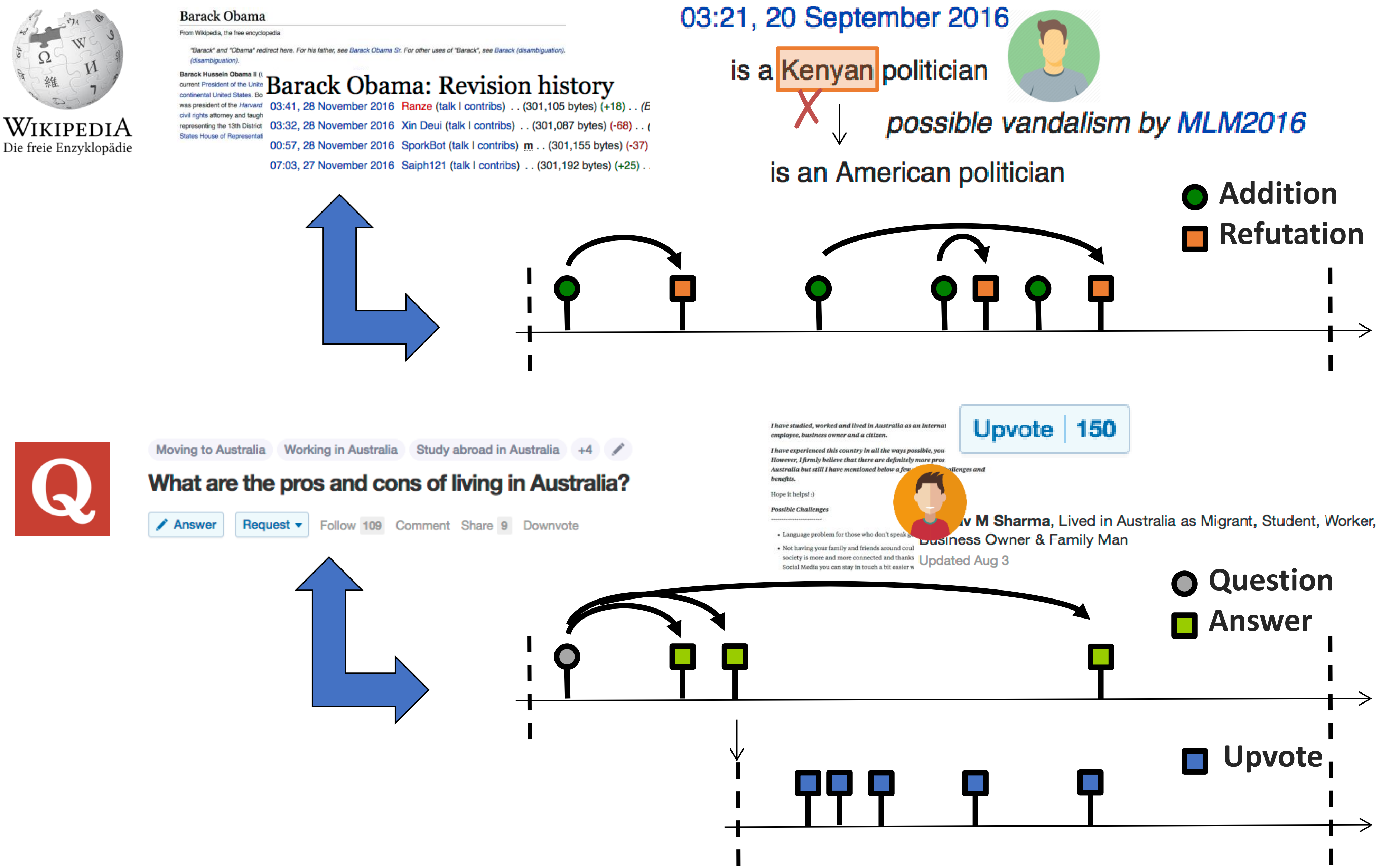


Friggeri et al., 2014

They can have an impact in the off-line world

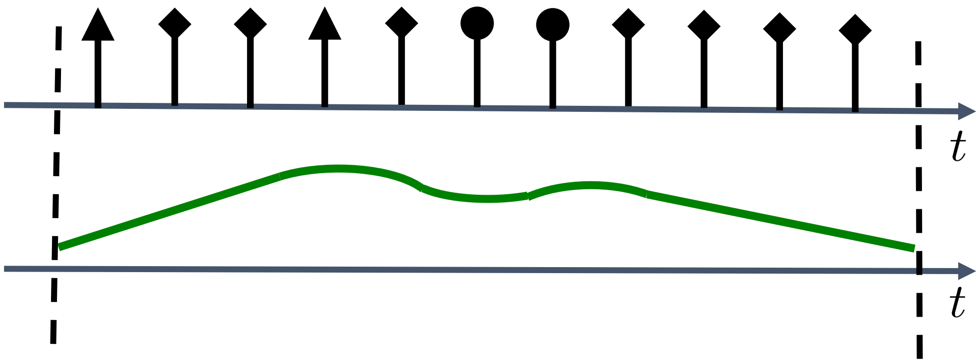
theguardian

Click and elect: how fake news helped Donald Trump win a real election

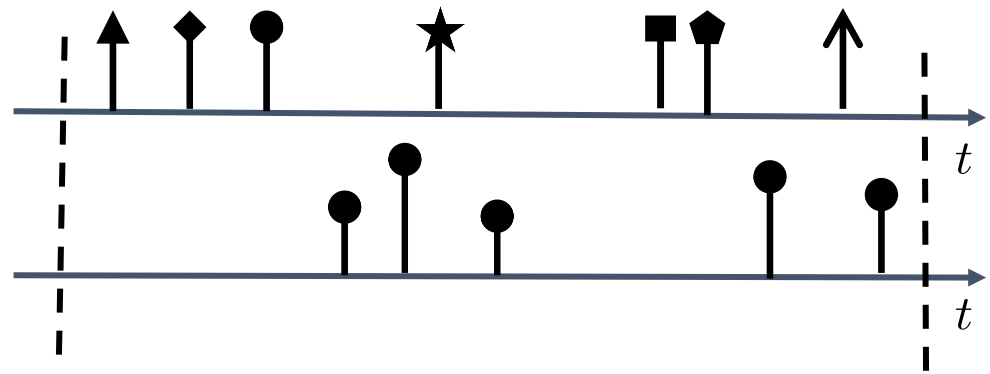


Wikipedia
Die freie Enzyklopädie
Barack Obama
From Wikipedia, the free encyclopedia
Barack Obama: Revision history
03:41, 28 November 2016 Ranze (talk | contribs) . . (301,105 bytes) (+18)
03:32, 28 November 2016 Xin Deui (talk | contribs) . . (301,087 bytes) (-68)
00:57, 28 November 2016 SporkBot (talk | contribs) m . . (301,155 bytes) (-37)
07:03, 27 November 2016 Saiph121 (talk | contribs) . . (301,192 bytes) (+25)
03:21, 20 September 2016
is a Kenyan politician
possible vandalism by MLM2016
is an American politician
Addition
Refutation
t
Moving to Australia
Working in Australia
Study abroad in Australia
+4
What are the pros and cons of living in Australia?
Answer
Request
Follow 109
Comment
Share 9
Downvote
Upvote 150
Business Owner & Family Man
Updated Aug 3
Question
Answer
t
Upvote
t

Aren’t these event traces just time series?

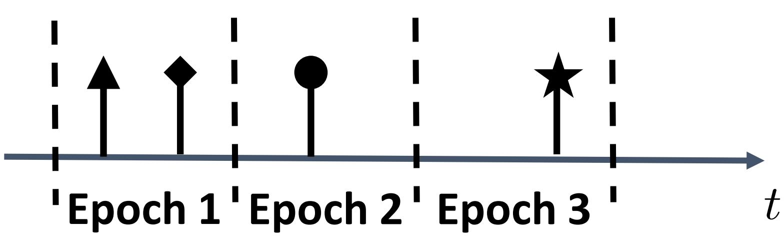


Discrete and continuous times series



Discrete events in continuous time

What about aggregating events in *epochs*?



How long is each epoch?

How to aggregate events per epoch?

What if no event in one epoch?

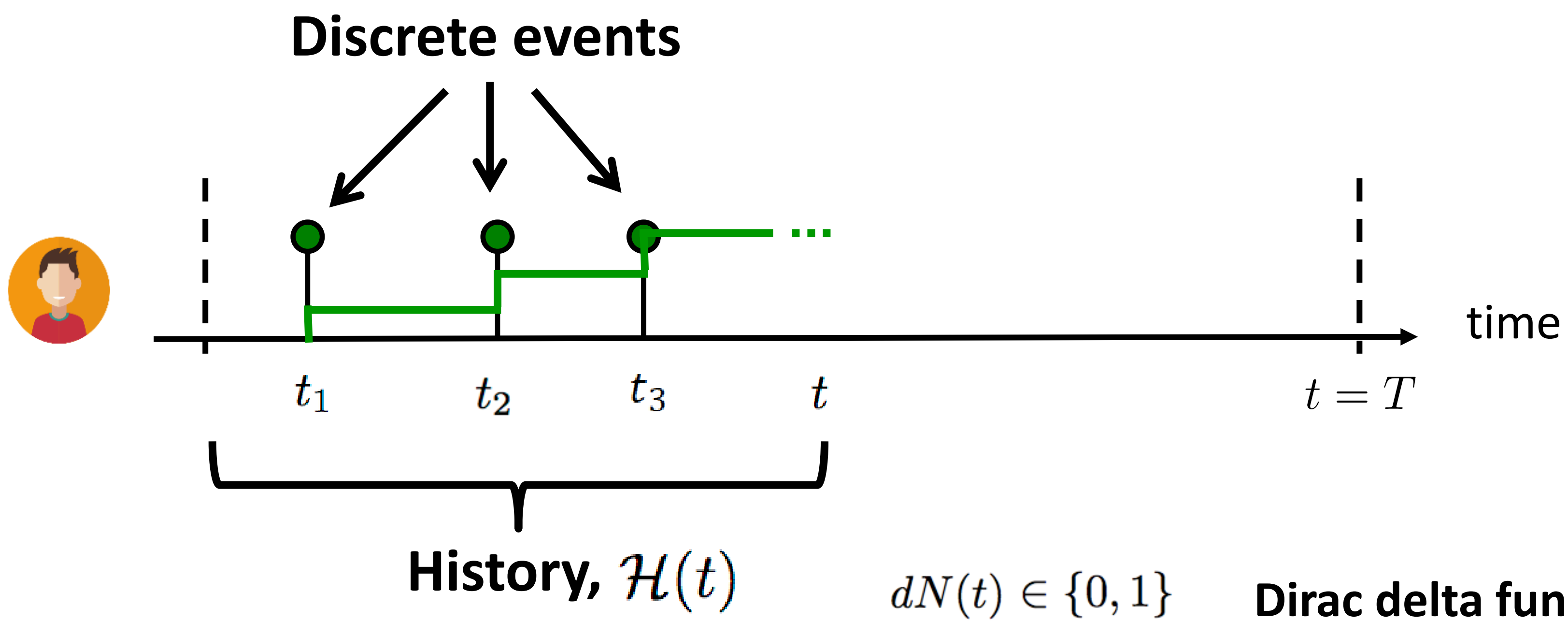
What about time-related queries?

Temporal Point Processes (TPPs):
Introduction

1. **Intensity function**
2. Basic building blocks
3. Superposition
4. Marks and SDEs with jumps

Temporal point processes

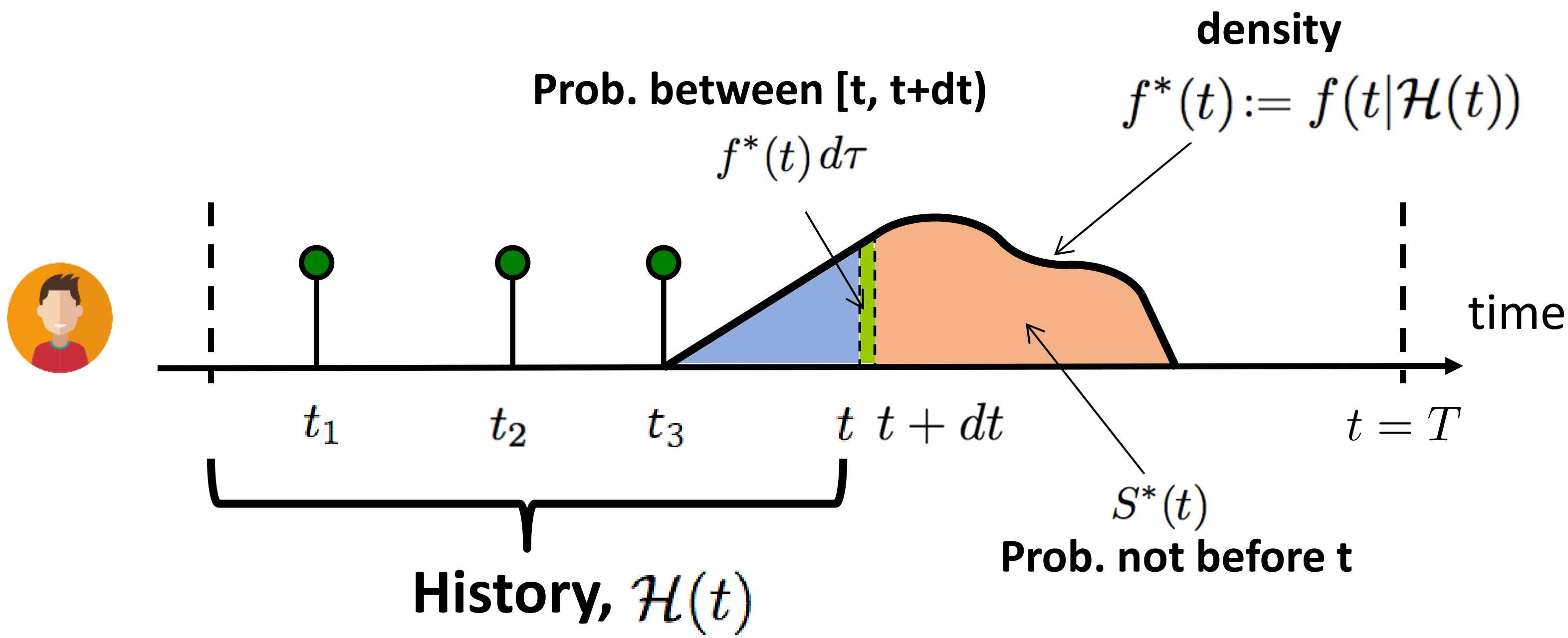
Temporal point process:
A random process whose realization consists of **discrete events localized in time** $\mathcal{H} = \{t_i\}$

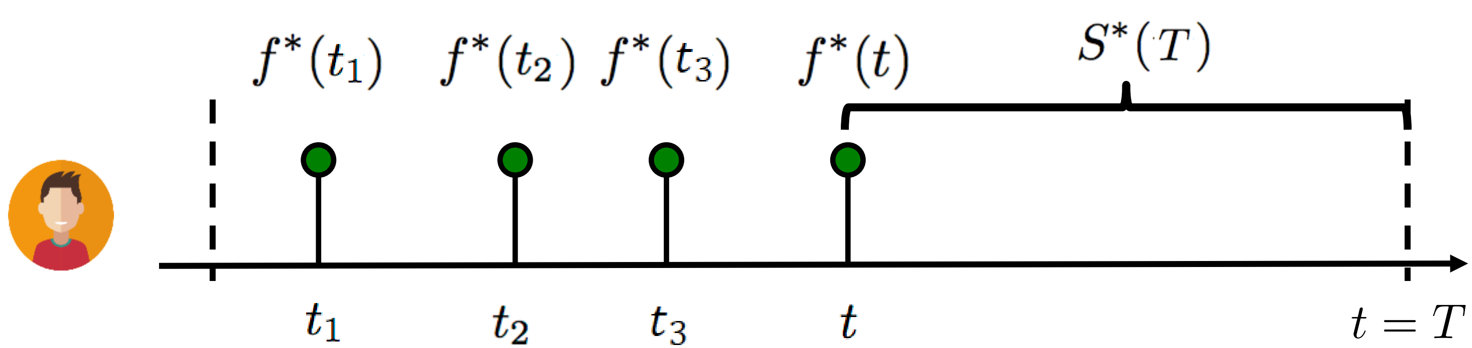


$dN(t) \in \{0, 1\}$ **Dirac delta function**

Formally: $N(t) = \int_0^t dN(s)$ ➡ $dN(t) = \sum_{t_i \in \mathcal{H}} \delta(t - t_i)\, dt$

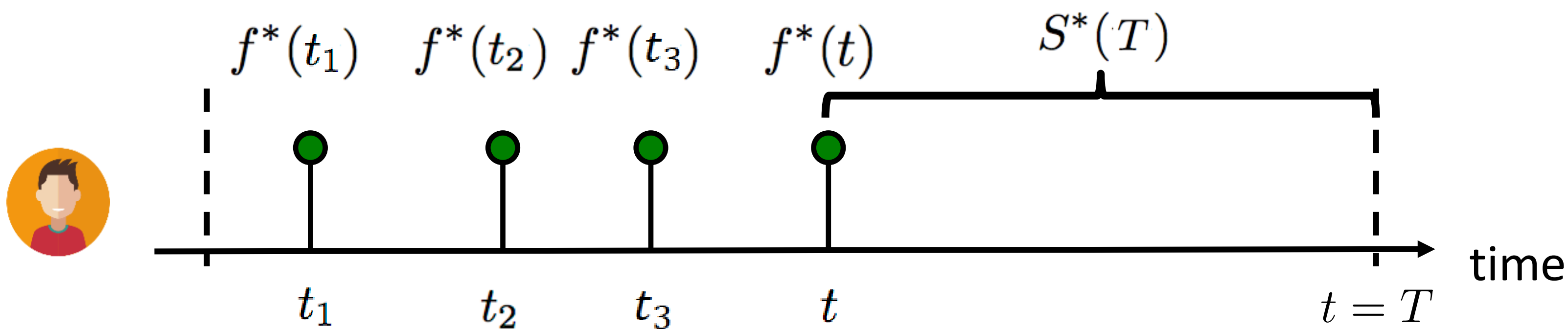
Model time as a random variable





Likelihood of a timeline: $f^*(t_1)\ f^*(t_2)\ f^*(t_3)\ f^*(t)\ S^*(T)$

Problems of density parametrization (I)



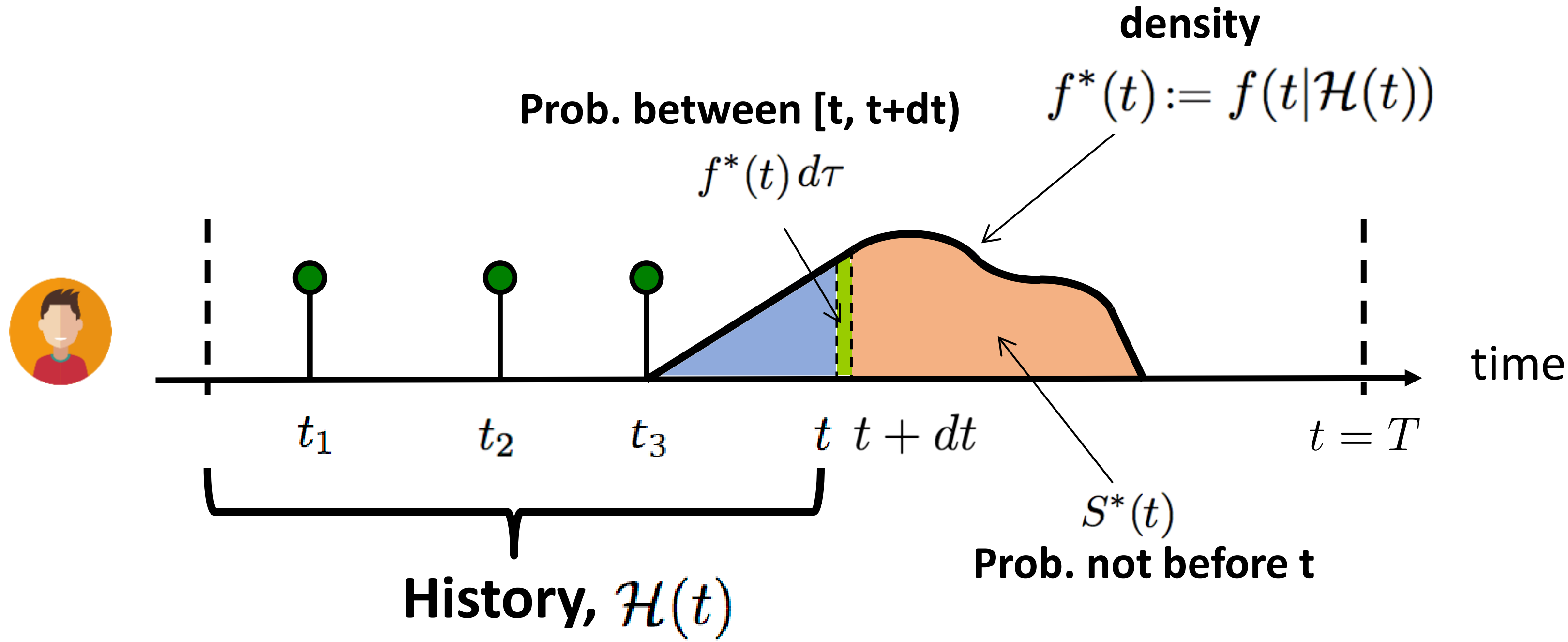
$$f^*(t_1) \quad f^*(t_2) \quad f^*(t_3) \quad f^*(t) \quad S^*(T)$$

$$\frac{\exp\langle w, \psi^*(t_1)\rangle}{Z} \quad \frac{\exp\langle w, \psi^*(t_2)\rangle}{Z} \quad \frac{\exp\langle w, \psi^*(t_3)\rangle}{Z} \quad \frac{\exp\langle w, \psi^*(t)\rangle}{Z} \quad 1 - \int_t^T \frac{\exp\langle w, \psi^*(\tau)\rangle}{Z} d\tau$$

It is difficult for model design and interpretability:

1. **Densities need to integrate to 1 (i.e., partition function)**
2. **Difficult to combine timelines**

Intensity function



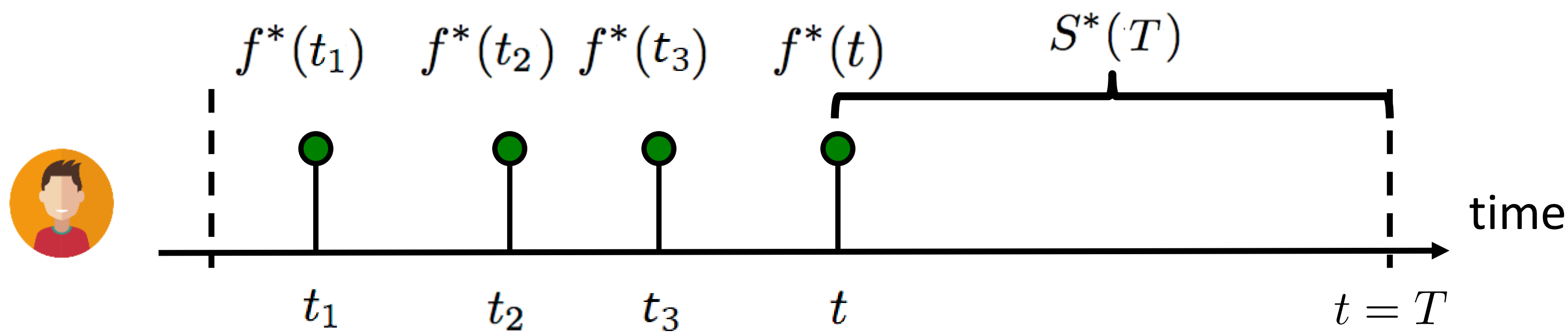
Intensity:

Probability between [t, t+dt) but not before t

$$\lambda^*(t)dt = \frac{f^*(t)dt}{S^*(t)} \geq 0 \quad \Rightarrow \quad \lambda^*(t)dt = \mathbb{E}[dN(t)|\mathcal{H}(t)]$$

Observation: $\lambda^*(t)$ **It is a rate = # of events / unit of time**

Advantages of intensity parametrization (I)



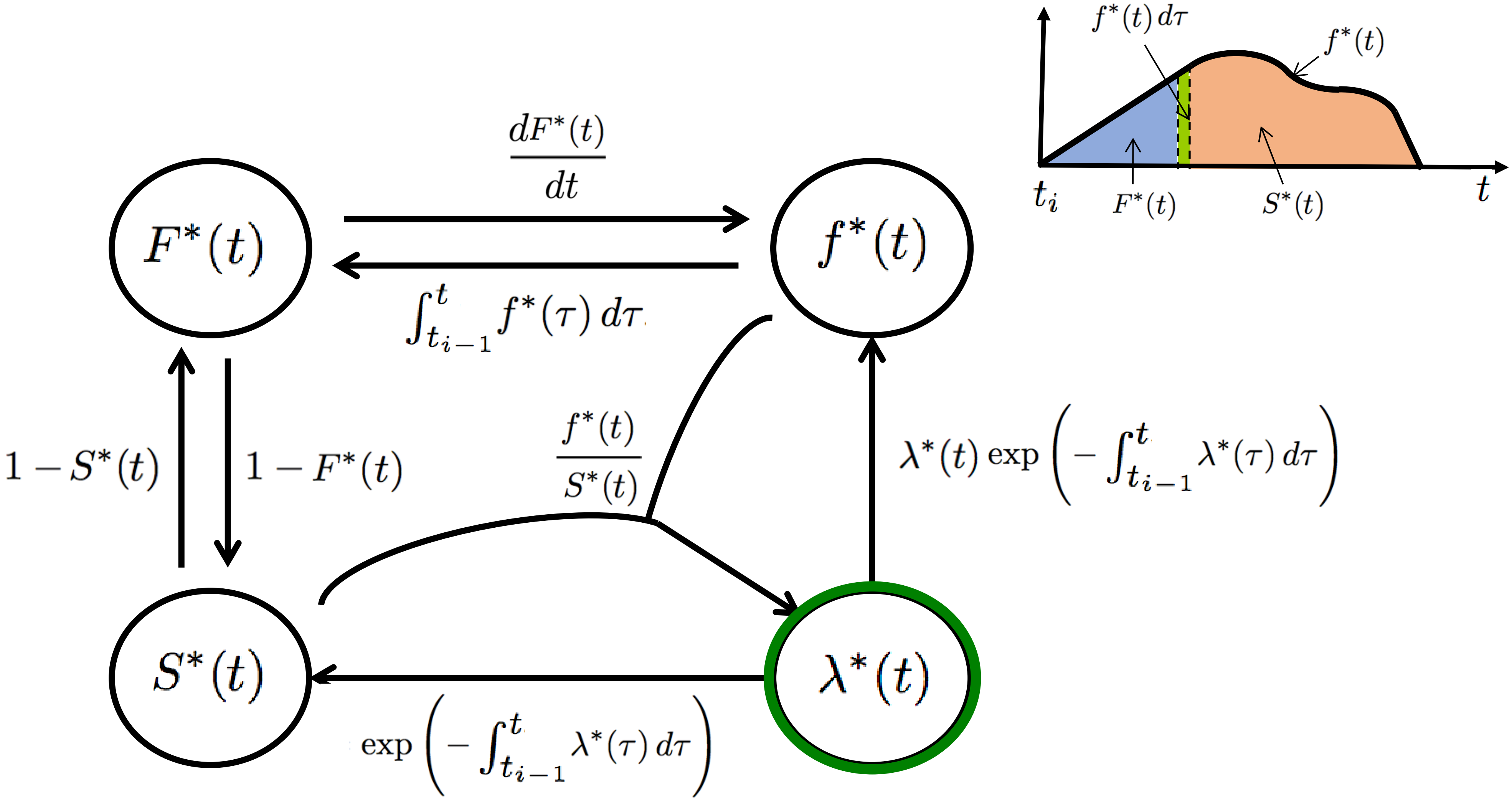
$$\lambda^*(t_1)\ \lambda^*(t_2)\ \lambda^*(t_3)\ \lambda^*(t)\ \exp\left(-\int_0^T \lambda^*(\tau)\, d\tau\right)$$

$$\langle w, \phi^*(t_1)\rangle \quad \langle w, \phi^*(t_2)\rangle \quad \langle w, \phi^*(t_3)\rangle \quad \langle w, \phi^*(t)\rangle \quad \exp\left(-\int_0^T \langle w, \phi^*(\tau)\rangle d\tau\right)$$

Suitable for model design and **interpretable**:

1. **Intensities only need to be nonnegative**
2. **Easy to combine timelines**

Relation between f*, F*, S*, λ*



Representation:
Temporal Point Processes

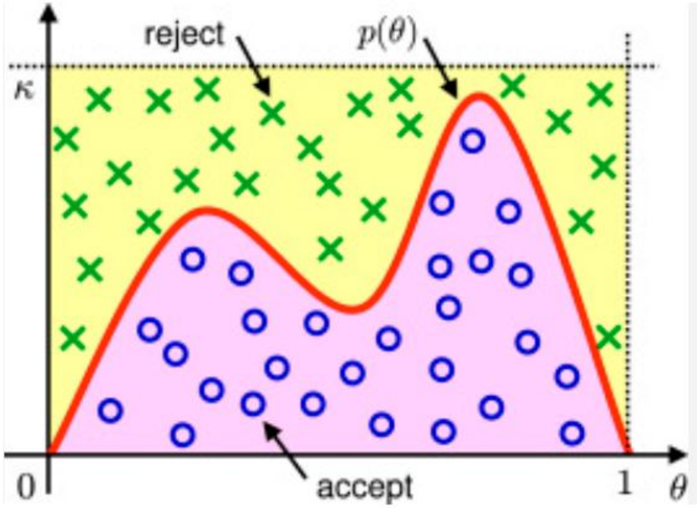
1. Intensity function
2. **Basic building blocks**
3. Superposition
4. Marks and SDEs with jumps

Recall: Some Sampling Techniques

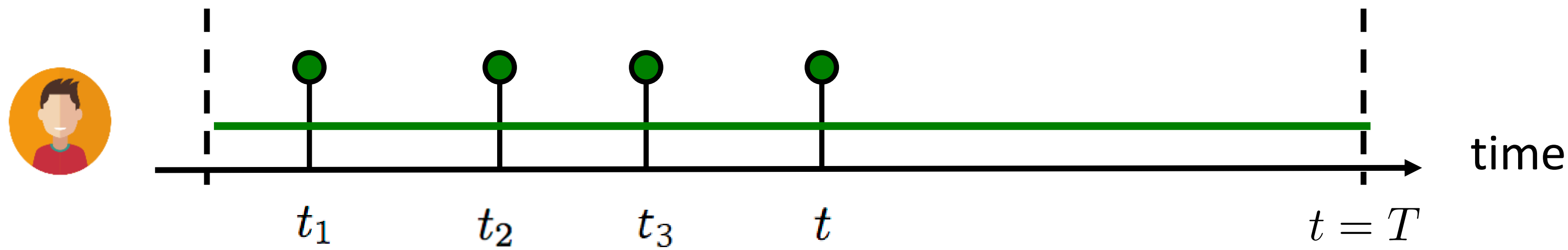
- Sampling is essential in statistics because it makes inference more efficient, feasible, accurate, and resource-effective while allowing for generalizability and detailed analysis.
- We treat sampling methods in more detail at the end of the course.
- **Inversion sampling**: Also known as inverse transform sampling, is a method for generating random samples from any probability distribution given its cumulative distribution function (CDF), in two steps:
  - Uniform Random Sample: Generate a random number (u) from a uniform distribution between 0 and 1.
  - Inverse CDF: Use the inverse of the cumulative distribution function (CDF) of the target distribution to transform the uniform random sample. This involves finding the value (x) such that (F(x) = u), where (F) is the CDF of the target distribution.

Recall: Some Sampling Techniques

- **Rejection sampling,** also known as the acceptance-rejection method, is a technique used in computational statistics to generate observations from a target distribution by using a proposal distribution:
  - Proposal Distribution: Choose a proposal distribution (P(x)) from which it is easy to sample. This distribution should cover the support of the target distribution (f(x)).
  - Sampling: Generate a samples (x) from the proposal distribution (g(x)).
  - Acceptance Criterion: Accept the sample (x) if the defined acceptance criterion is met. Repeat the process until a sample is accepted.



Poisson process



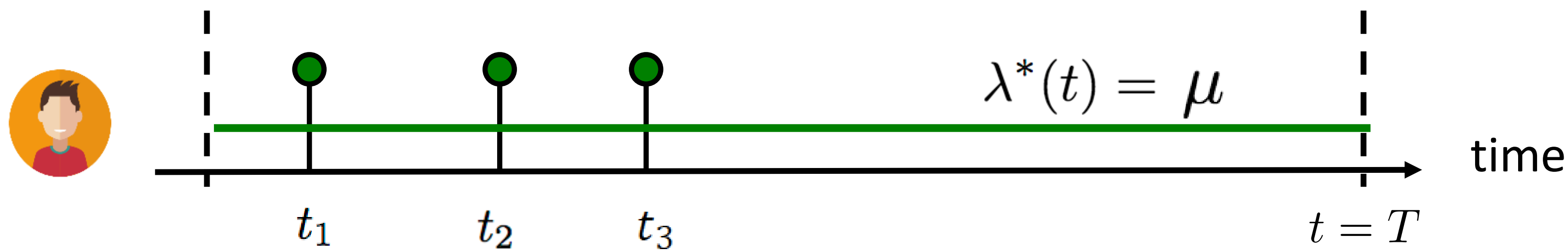
Intensity of a Poisson process

$$\lambda^*(t) = \mu$$

Observations:

1. **Intensity independent of history**
2. **Uniformly random occurrence**
3. **Time interval follows exponential distribution**

Fitting & sampling from a Poisson



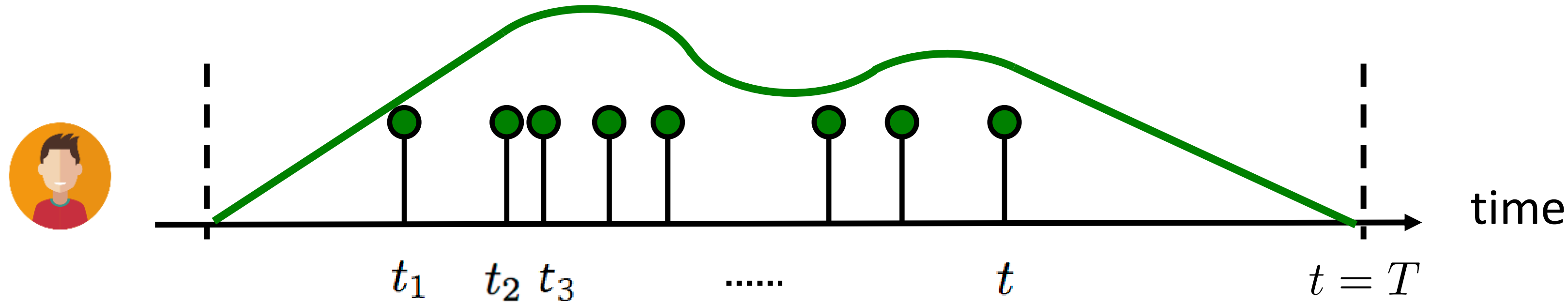
Fitting by maximum likelihood:

$$\mu^* = \underset{\mu}{\operatorname{argmax}} \; 3\log\mu - \mu T = \frac{3}{T}$$

Sampling using inversion sampling:

$$t \sim \underbrace{\mu \exp(-\mu(t - t_3))}_{f_t^*(t)} \;\Rightarrow\; t = \underbrace{-\frac{1}{\mu}\log(1 - \overset{Uniform(0,1)}{\overset{\downarrow}{u}}) + t_3}_{F_t^{-1}(u)}$$

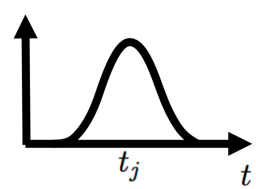
Inhomogeneous Poisson process



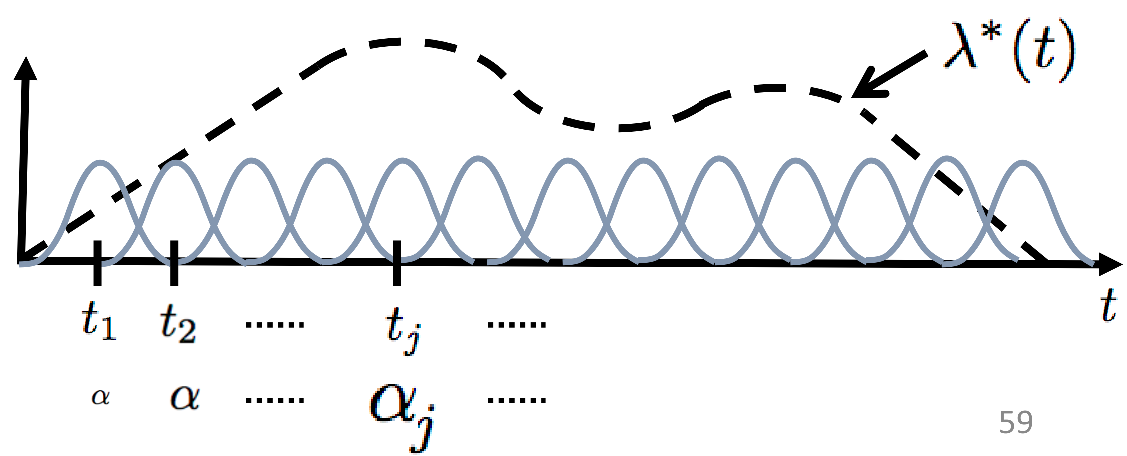
Intensity of an inhomogeneous Poisson process

$$\lambda^*(t) = g(t) \geqslant 0$$ **(Independent of history)**

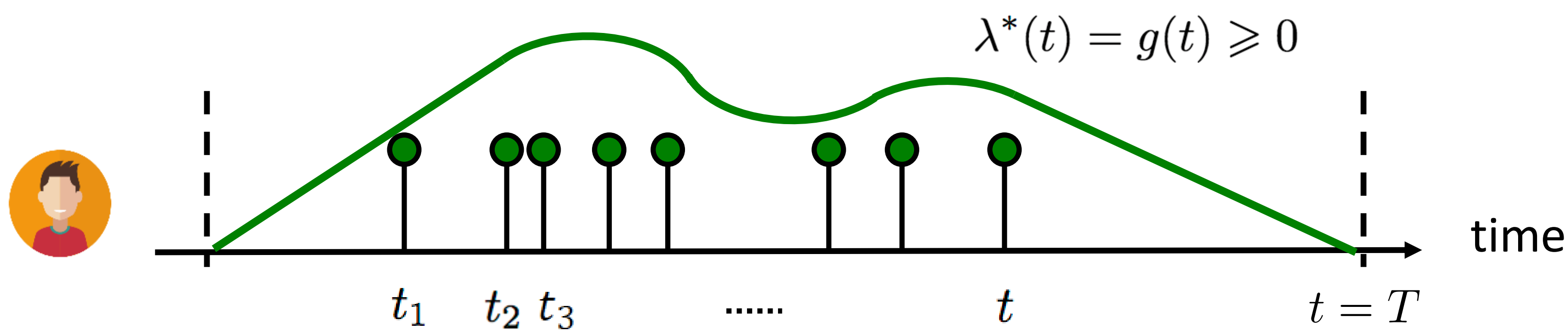
Example:



$$\lambda^*(t) = \sum_j \alpha_j \overbrace{k(t - t_j)}$$



Fitting & sampling from inhomogeneous Poisson



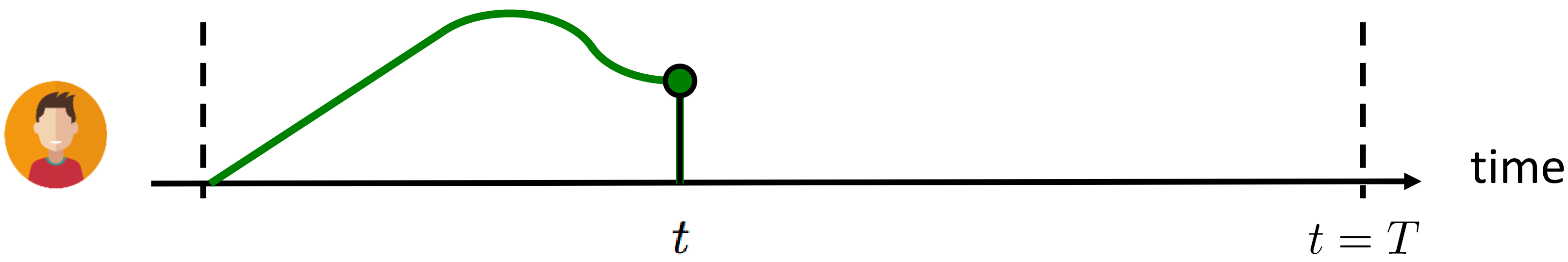
Fitting by maximum likelihood: $\underset{g(t)}{\text{maximize}} \sum_{i=1}^{n} \log g(t_i) - \int_0^T g(\tau)\, d\tau$

Sampling using thinning (reject. sampling) + inverse sampling:

1. **Sample** t **from Poisson process with intensity** μ **using inverse sampling**
2. **Generate** $u_2 \sim Uniform(0,1)$
3. **Keep the sample if** $u_2 \leq g(t) / \mu$

Keep sample with prob. $g(t)/\mu$

Terminating (or survival) process



Intensity of a terminating (or survival) process

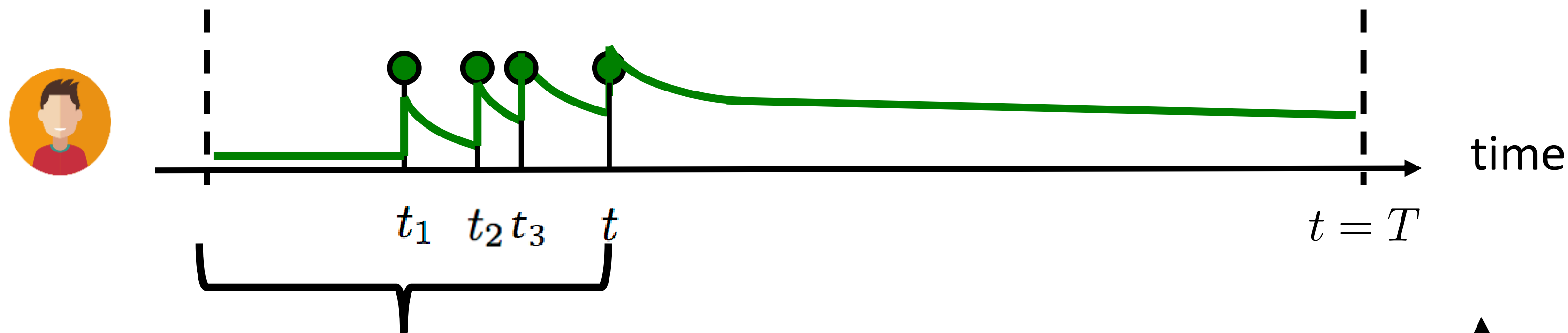
$$\lambda^*(t) = g^*(t)(1 - N(t)) \geqslant 0$$

Observations:

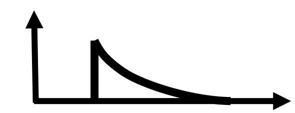
1. **Limited number of occurrences**

Try sampling and fitting!

Self-exciting (or Hawkes) process



History, $\mathcal{H}(t)$



Intensity of self-exciting (or Hawkes) process:

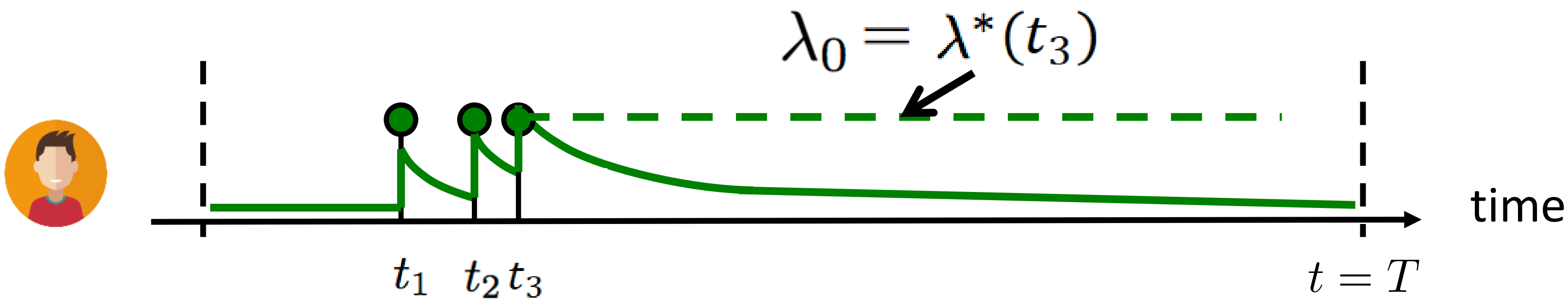
$$\lambda^*(t) = \mu + \alpha \sum_{t_i \in \mathcal{H}(t)} \overbrace{\kappa_\omega(t - t_i)}^{\textbf{Triggering kernel}}$$

$$= \mu + \alpha \kappa_\omega(t) \star dN(t)$$

Observations:

1. **Clustered (or bursty) occurrence of events**
2. **Intensity is stochastic and history dependent**

Fitting a Hawkes process from a recorded timeline



Fitting by maximum likelihood:

$$\underset{\mu,\alpha}{\text{maximize}} \sum_{i=1}^{n} \log \lambda^*(t_i) - \int_0^T \lambda^*(\tau)\, d\tau$$

The max. likelihood is jointly convex in μ and α

Sampling using thinning (reject. sampling) + inverse sampling:

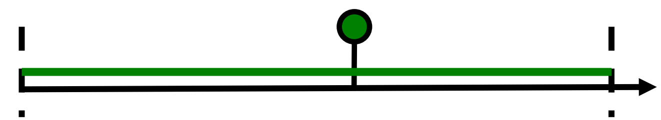
Key idea: the maximum of the intensity λ_0 changes over time

Summary

Building blocks to represent **different dynamic processes:**

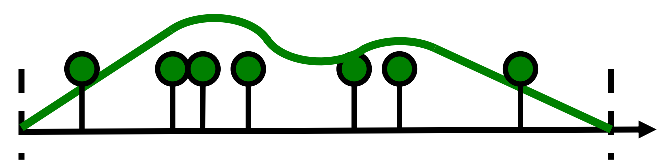
Poisson processes:

$$\lambda^*(t) = \lambda$$



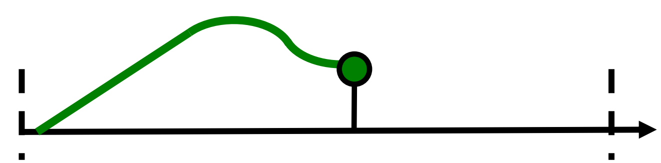
Inhomogeneous Poisson processes:

$$\lambda^*(t) = g(t)$$



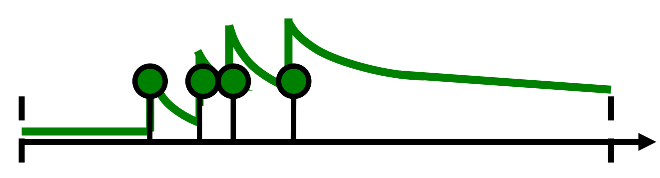
Terminating point processes:

$$\lambda^*(t) = g^*(t)(1 - N(t))$$



Self-exciting point processes:

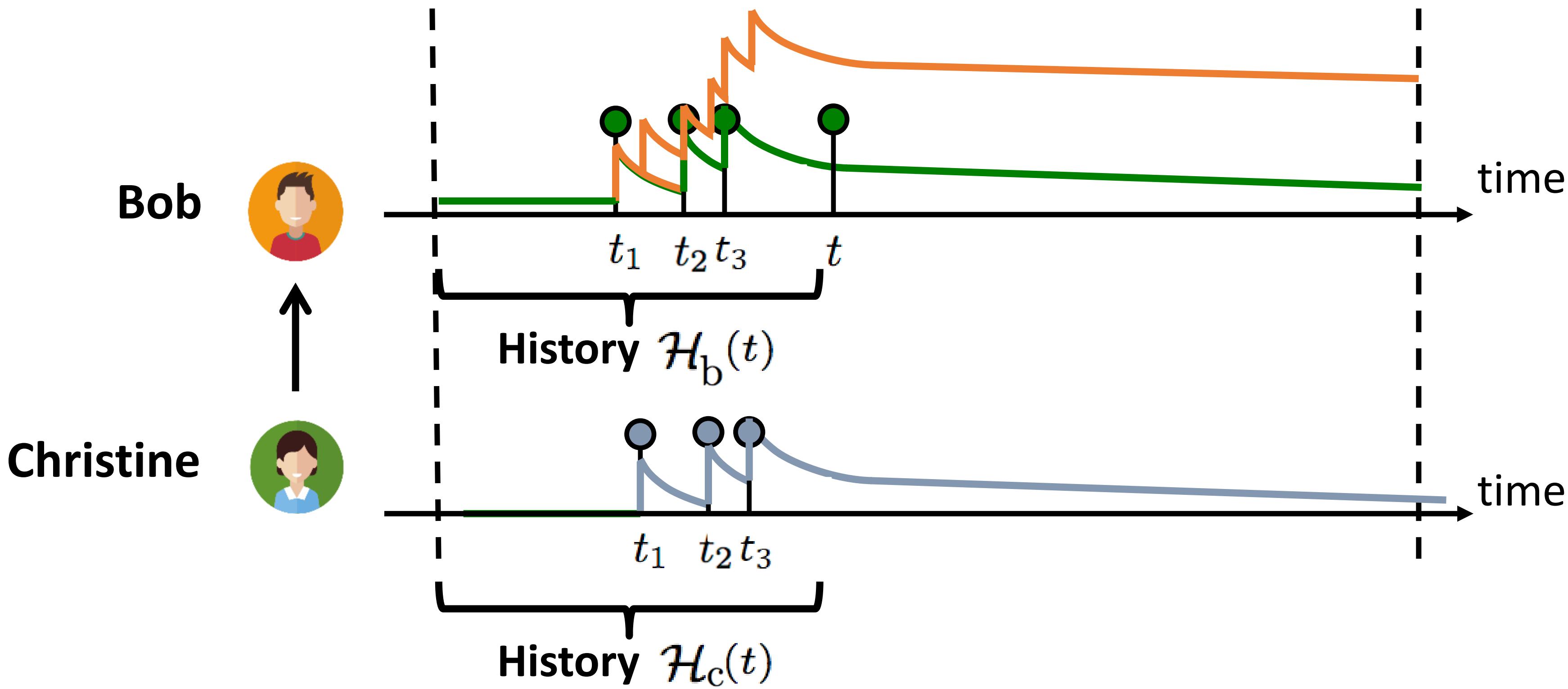
$$\lambda^*(t) = \mu + \alpha \sum_{t_i \in \mathcal{H}(t)} \kappa_\omega(t - t_i)$$



Representation:
Temporal Point Processes

1. Intensity function
2. Basic building blocks
3. **Superposition**
4. Marks and SDEs with jumps

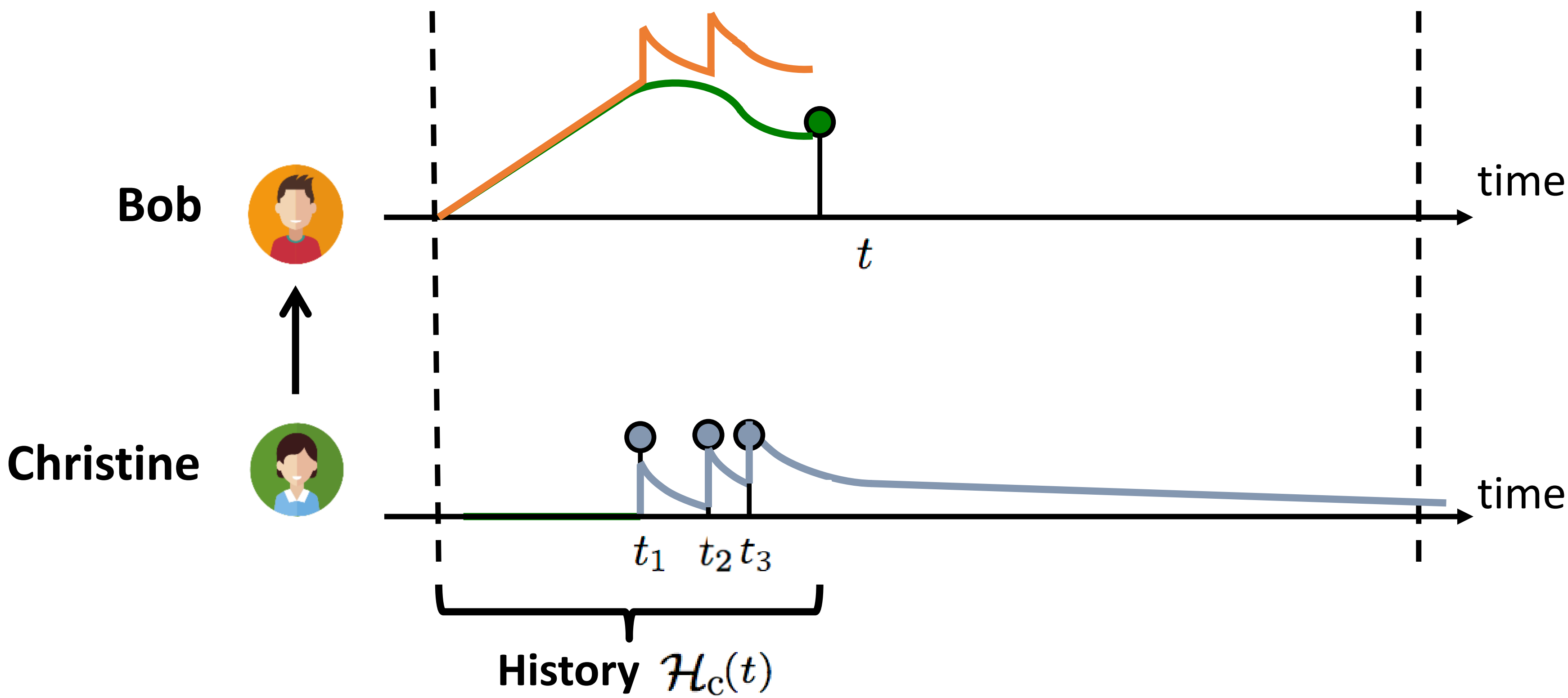
Mutually exciting process



Clustered occurrence affected by neighbors

$$\lambda^*(t) = \mu + \alpha \sum_{t_i \in \mathcal{H}_{\mathrm{b}}(t)} \kappa_\omega(t - t_i) + \beta \sum_{t_i \in \mathcal{H}_{\mathrm{c}}(t)} \kappa_\omega(t - t_i)$$

Mutually exciting terminating process



Clustered occurrence affected by neighbors

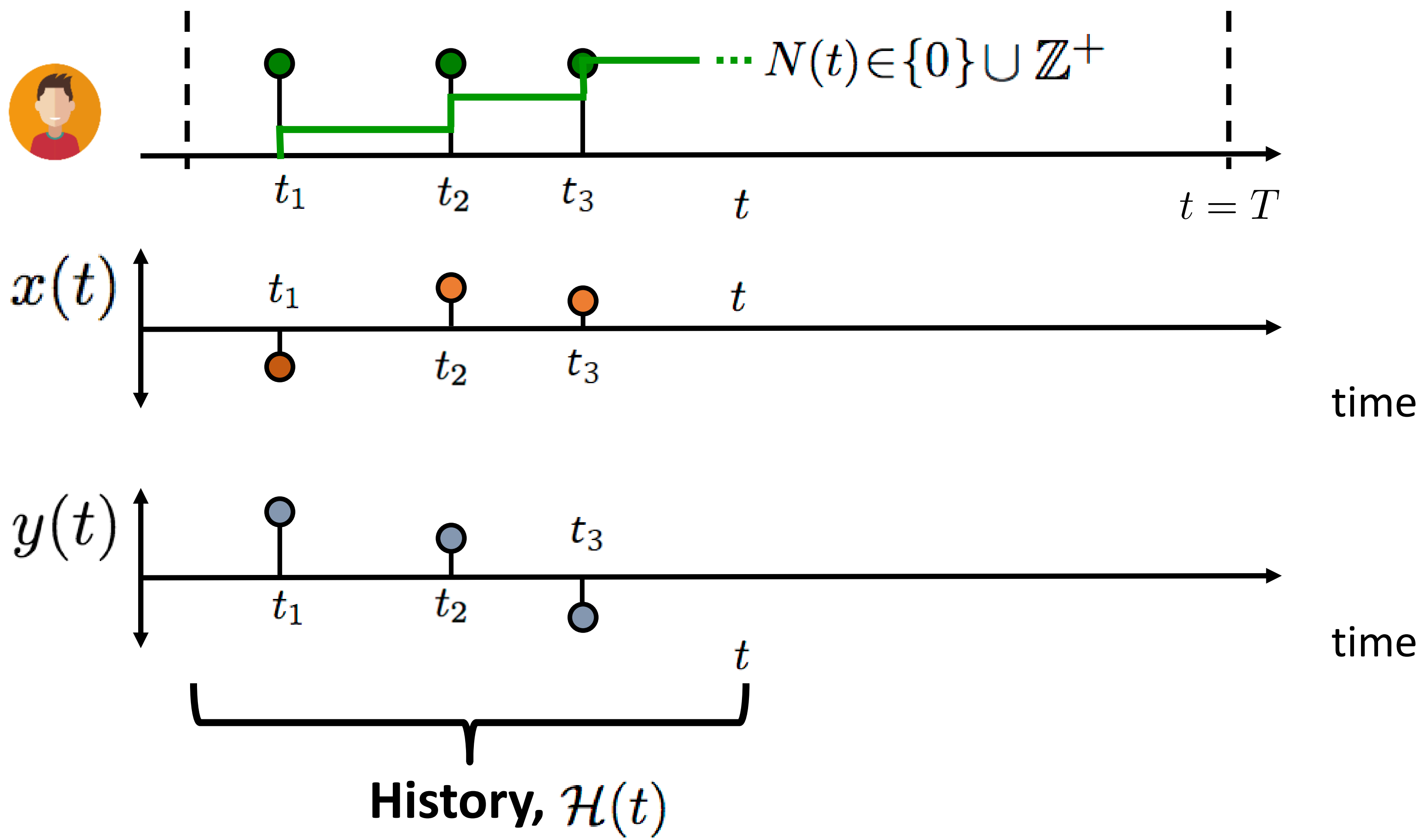
$$\lambda^*(t) = (1 - N(t))\Big(g(t) + \beta \sum_{t_i \in \mathcal{H}_c(t)} \kappa_\omega(t - t_i)\Big)$$

Representation:
Temporal Point Processes

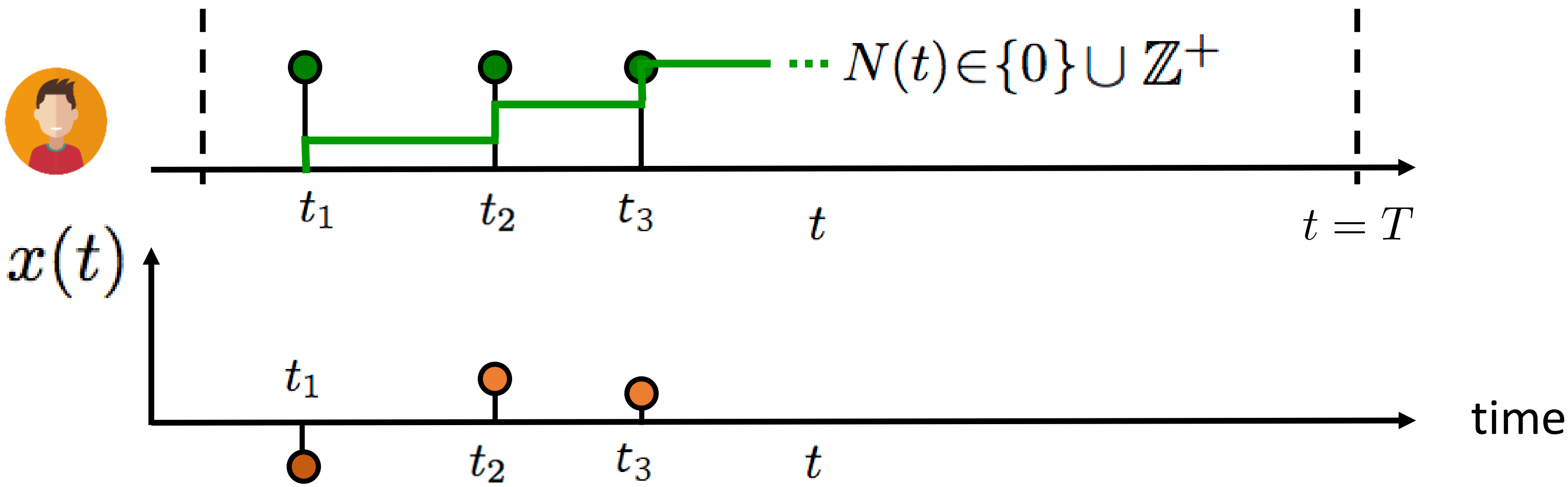
1. Intensity function
2. Basic building blocks
3. Superposition
4. Marks and SDEs with jumps

Marked temporal point processes

Marked temporal point process:
A random process whose realization consists of **discrete *marked* events localized in time**



Independent identically distributed marks



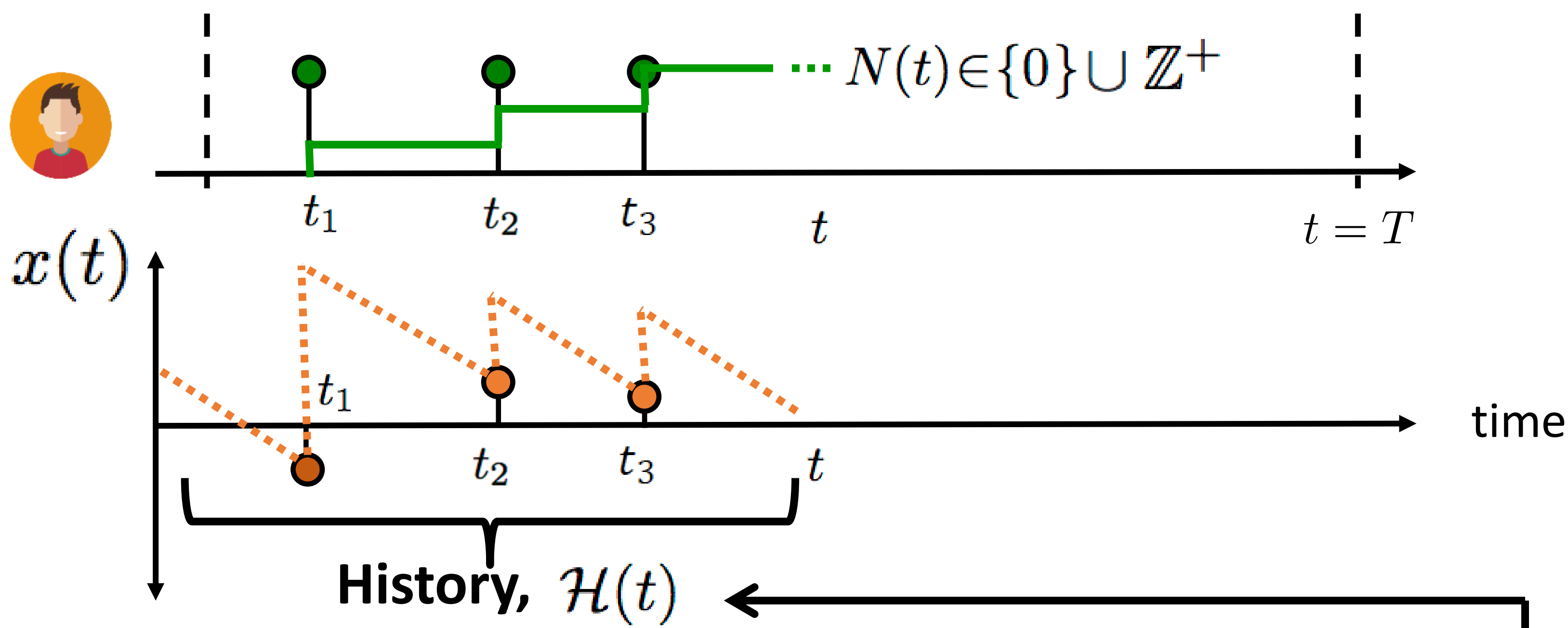
Distribution for the marks:

$$x^*(t_i) \sim p(x)$$

Observations:

1. **Marks independent of the temporal dynamics**
2. **Independent identically distributed (I.I.D.)**

Dependent marks: SDEs with jumps



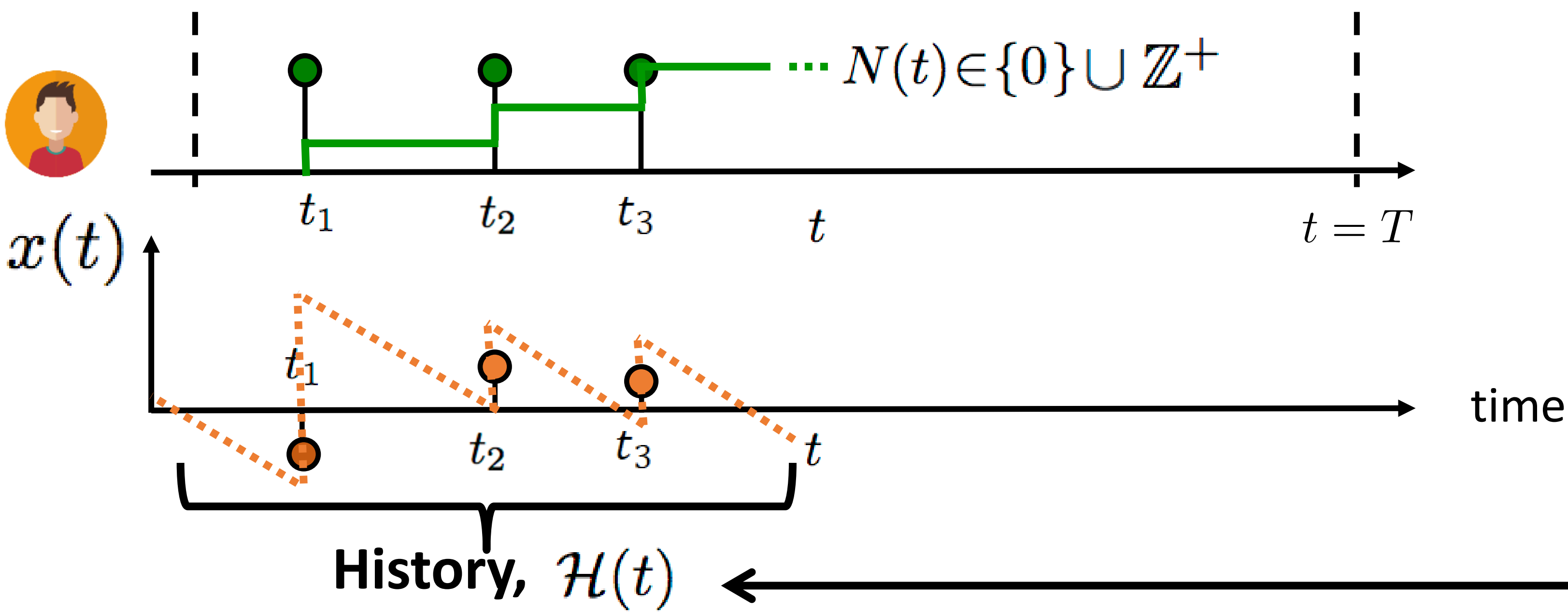
Marks given by stochastic differential equation with jumps:

$$x(t+dt) - x(t) = dx(t) = \underbrace{f(x(t),t)dt}_{\textbf{Drift}} + \underbrace{h(x(t),t)dN(t)}_{\textbf{Event influence}}$$

Observations:

1. **Marks dependent of the temporal dynamics**
2. **Defined for all values of t**

Dependent marks: distribution + SDE with jumps



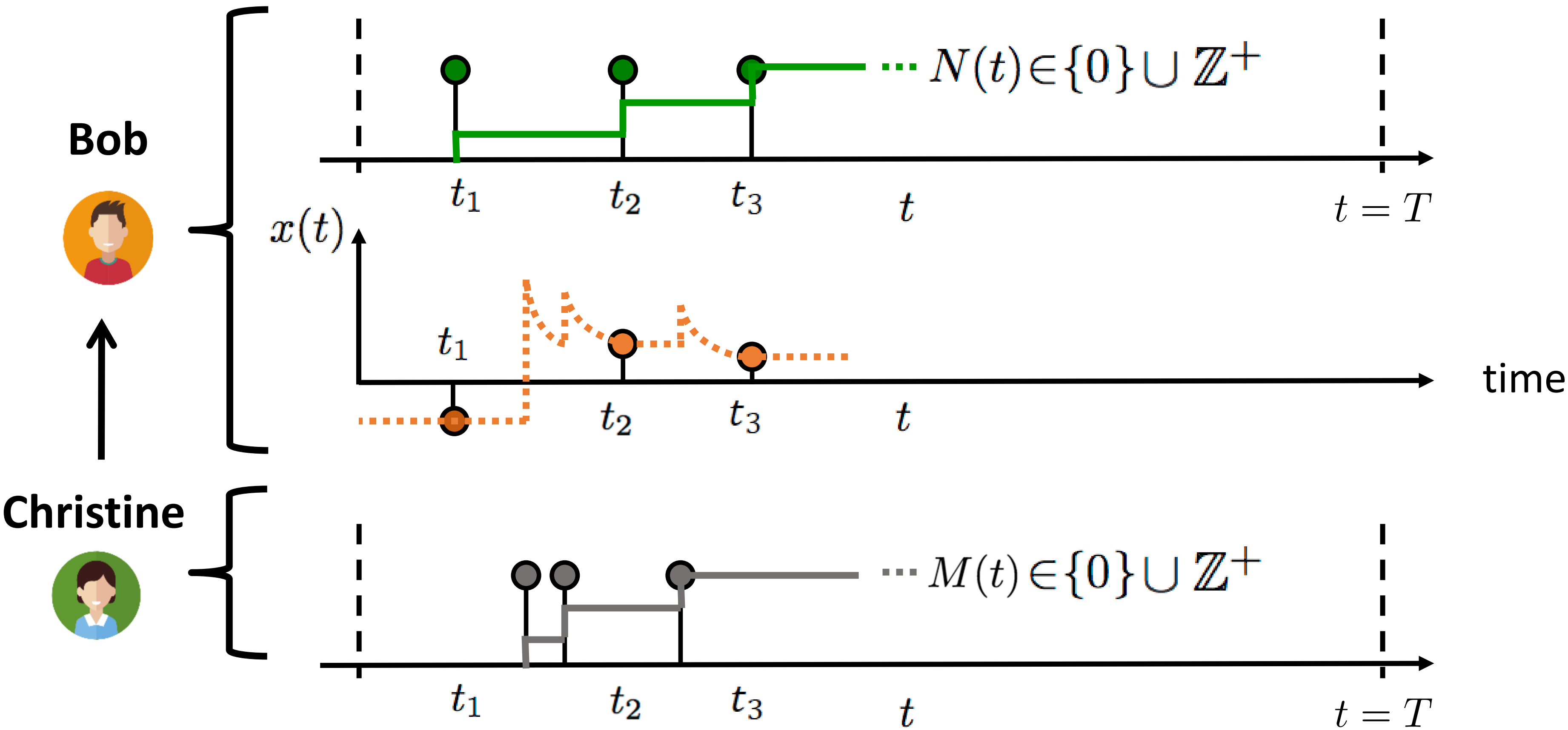
Distribution for the marks:

$$x^*(t_i) \sim p(x^* \,|\, x(t)) \;\Rightarrow\; dx(t) = \underbrace{f(x(t),t)dt}_{\text{Drift}} + \underbrace{h(x(t),t)dN(t)}_{\text{Event influence}}$$

Observations:

1. **Marks dependent on the temporal dynamics**
2. **Distribution represents additional source of uncertainty**

Mutually exciting + marks



Marks affected by neighbors

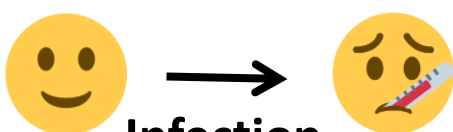
$$dx(t) = \underbrace{f(x(t), t)dt}_{\textbf{Drift}} + \underbrace{g(x(t), t)dM(t)}_{\textbf{Neighbor influence}}$$

Marked TPPs as stochastic dynamical systems

Example: Susceptible-Infected-Susceptible (SIS)

$X_i(t) = 0$ **Susceptible** $X_i(t) = 1$ **Infected** $X_i(t) = 0$ **Susceptible**

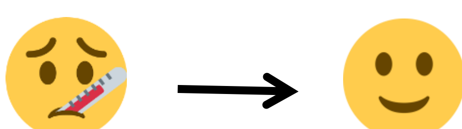
$$\overbrace{dX_i(t) = \underbrace{dY_i(t)}_{\text{It gets infected}} - \underbrace{dW_i(t)}_{\text{It recovers}}}^{\text{SDE with jumps}}$$



Infection rate

$$\mathbb{E}\left[dY_i(t)\right] = \lambda_{Y_i}(t)dt$$

$$\lambda_{Y_i}(t)dt = \overbrace{(1 - X_i(t))}^{\text{Node is susceptible}}\beta \underbrace{\textstyle\sum_{j \in \mathcal{N}(i)} X_j(t)dt}_{\text{If friends are infected, higher infection rate}}$$



Recovery rate

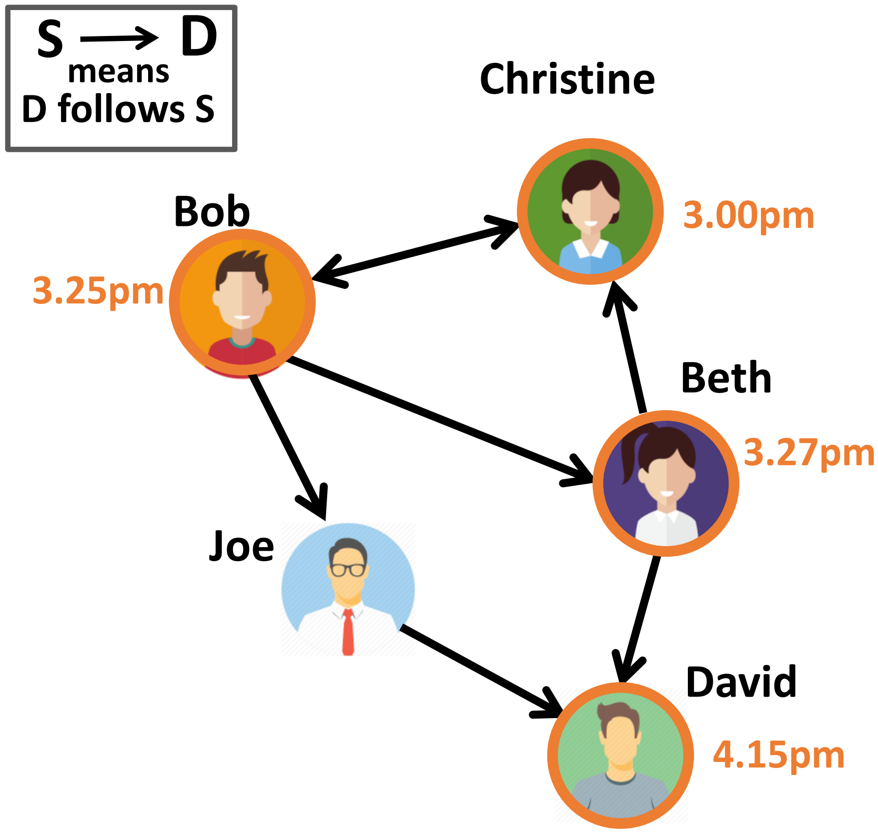
$$\mathbb{E}\left[dW_i(t)\right] = \lambda_{W_i}(t)dt$$

$$\overbrace{d\lambda_{W_i}(t) = \underbrace{\delta\, dY_i(t)}_{\text{Self-recovery rate when node gets infected}} \underbrace{- \lambda_{W_i}(t)dW_i(t)}_{\text{If node recovers, rate to zero}} + \underbrace{\rho\, dN_i(t)}_{\text{Rate increases if node gets treated}}}^{\text{SDE with jumps}}$$

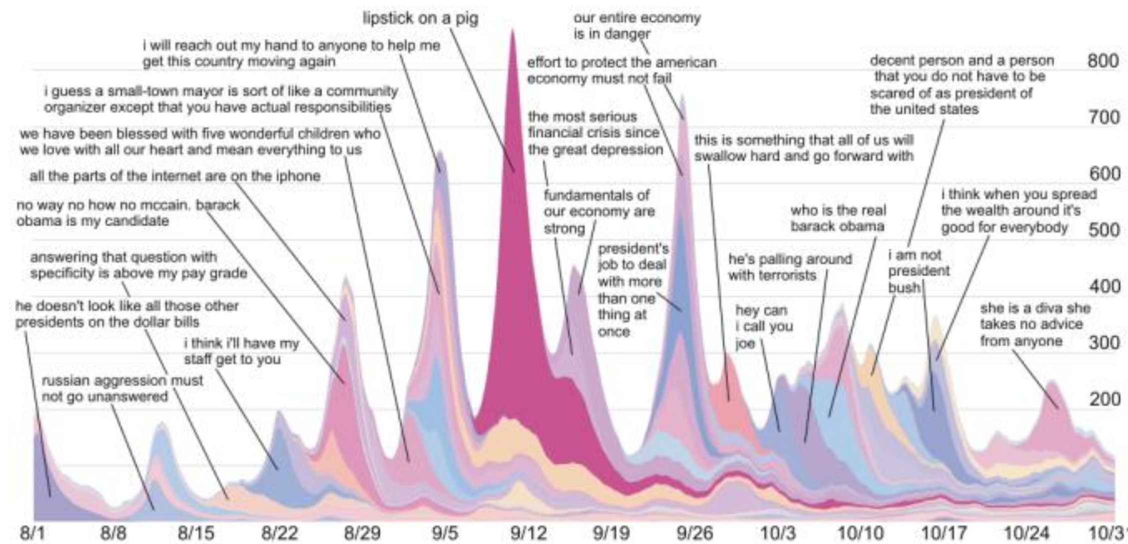
Models & Inference

1. **Modeling event sequences**
2. Clustering event sequences
3. Capturing complex dynamics
4. Causal reasoning on event sequences

Event sequences as cascades

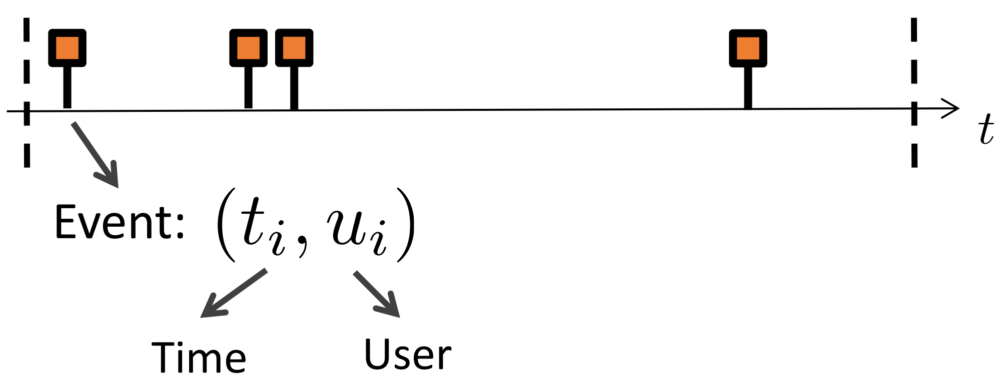


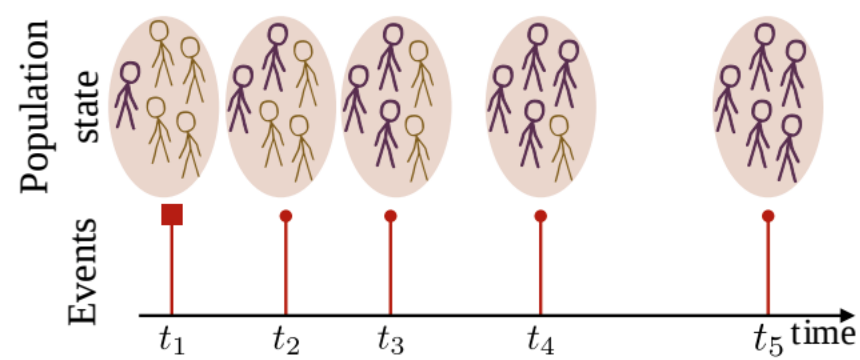
Information Diffusion



[Leskovec et al., 2009]

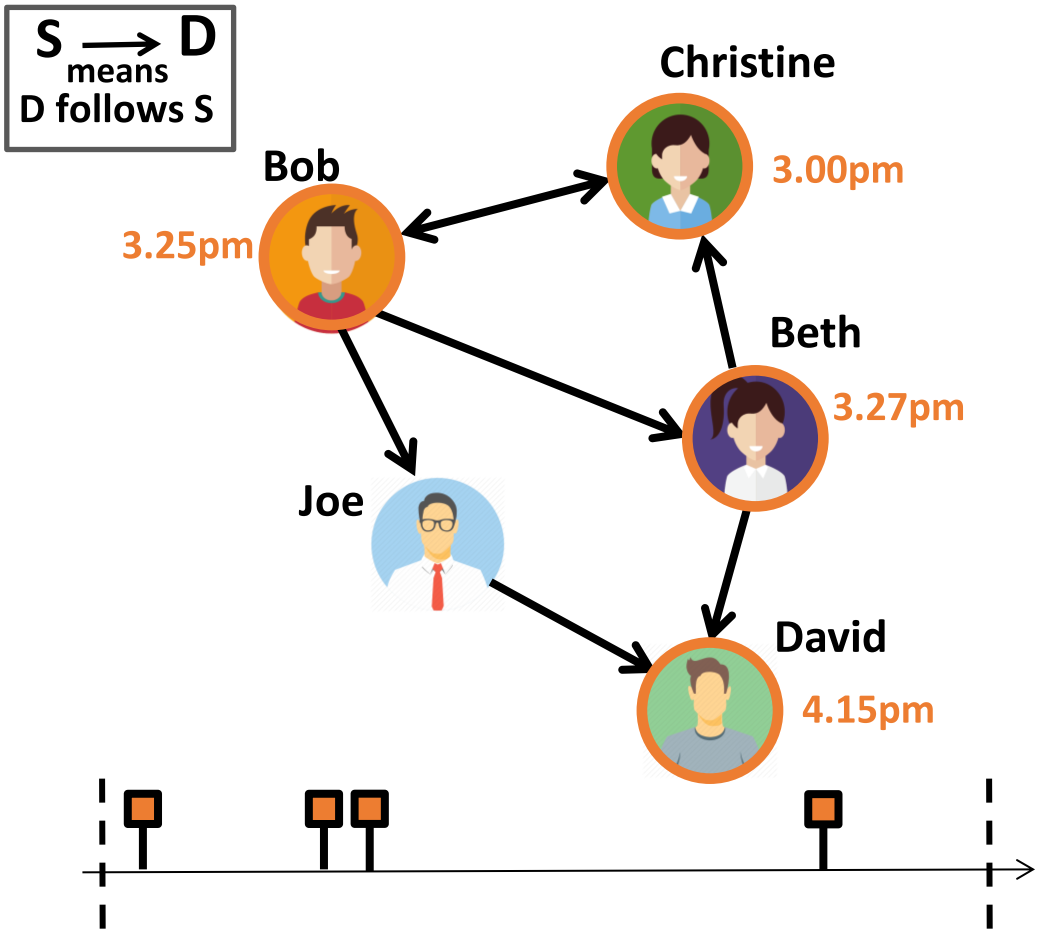
Disease Diffusion

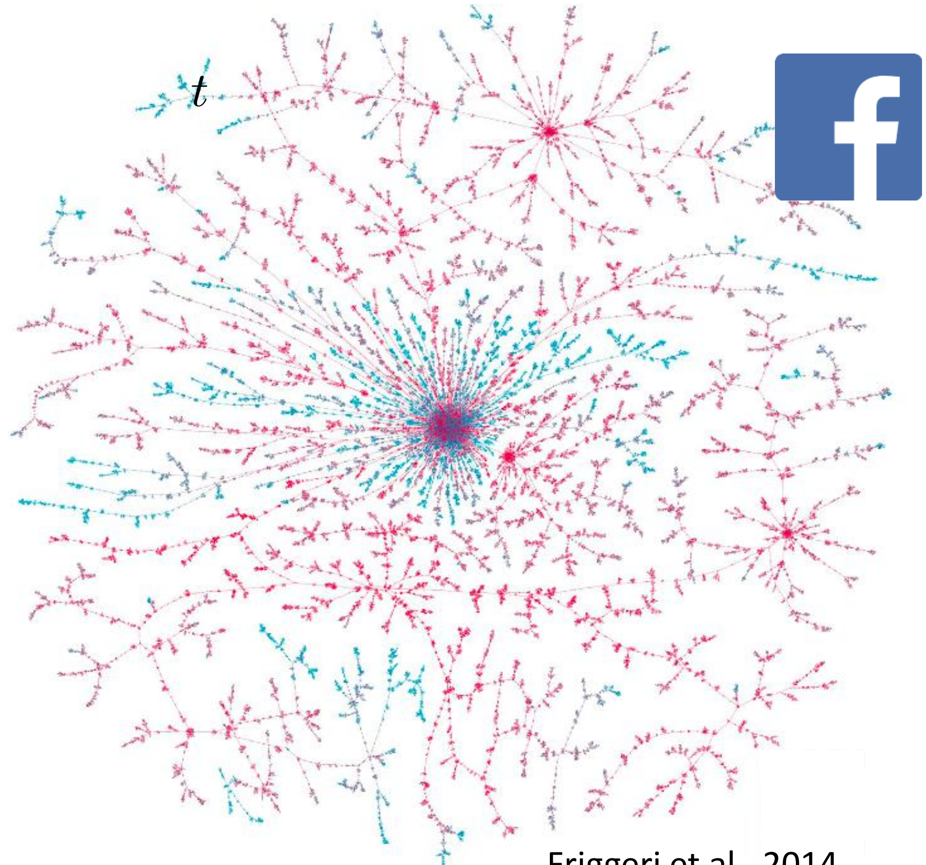




[Rizoiu et al., 2018]

An example: idea adoption





Friggeri et al., 2014

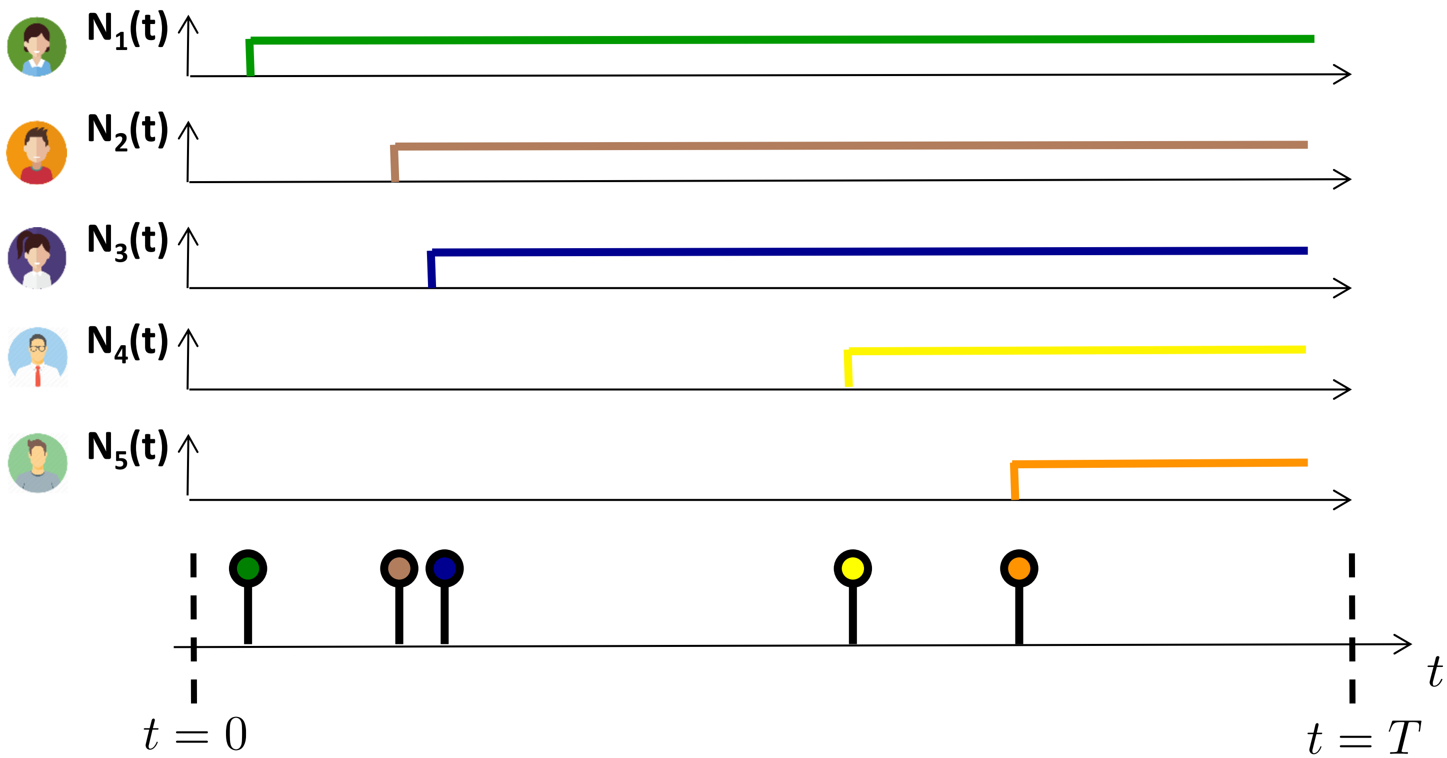
They can have an impact in the off-line world

theguardian

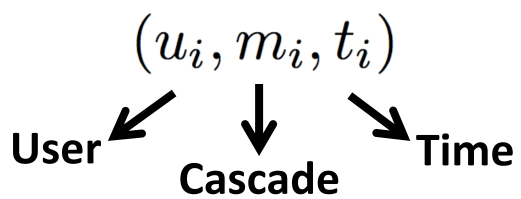
Click and elect: how fake news helped Donald Trump win a real election

Infection cascade representation

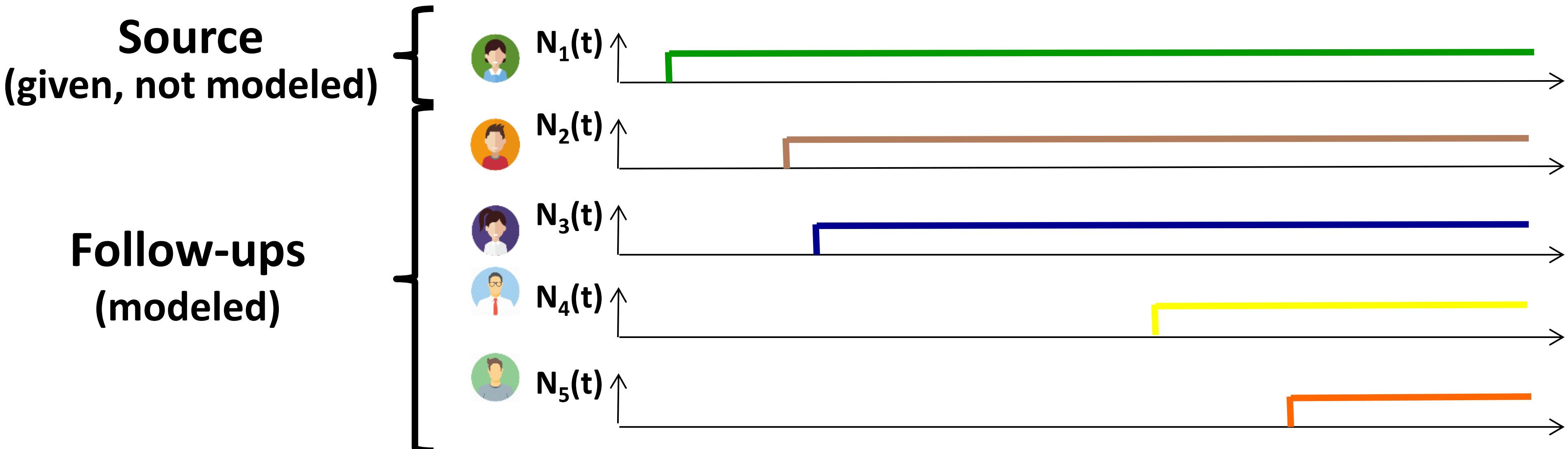
We represent an infection cascade using **terminating temporal point processes**:

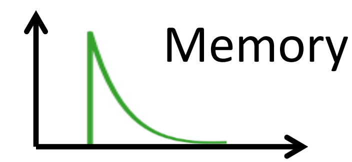


Infection event:



Infection intensity





$$\lambda_u^*(t) = \underbrace{(1 - N_u(t))}_{\text{Users get infected only once}} \sum_{v\in[m]} b_{vu} \underbrace{\sum_{e_i\in\mathcal{H}_v(t)} \kappa(t-t_i)}_{\text{Previous infections of user v}}$$

Influence from user v on user u

[Gomez-Rodriguez et al., ICML 2011]

Model inference from multiple cascades

Conditional intensities

$$\lambda_u^*(t)$$

Diffusion log-likelihood

$$\mathfrak{L} = \sum_{u=1}^{n} \log \lambda_u^*(t_u) - \int_0^T \lambda_u^*(\tau)\, d\tau$$

Sum up log-likelihoods of multiple cascades!

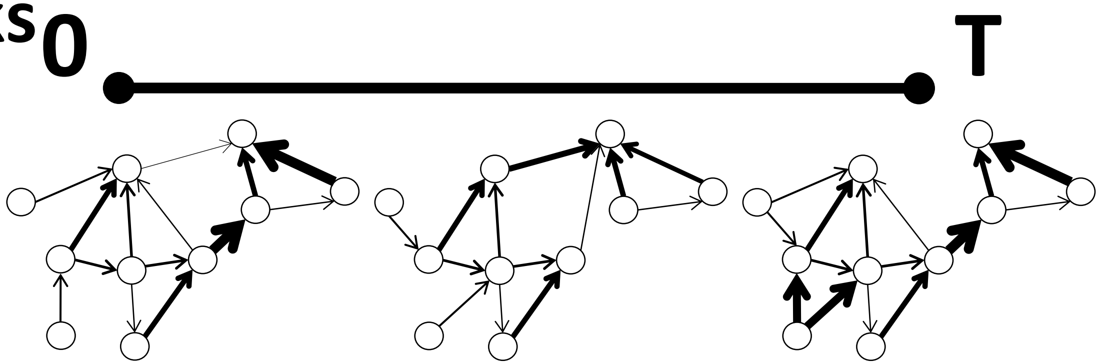
Maximum likelihood approach to find model parameters!

Theorem. For any choice of parametric memory, the **maximum likelihood** problem is **convex**.

[Gomez-Rodriguez et al., ICML 2011]

In some cases, influence change over time:

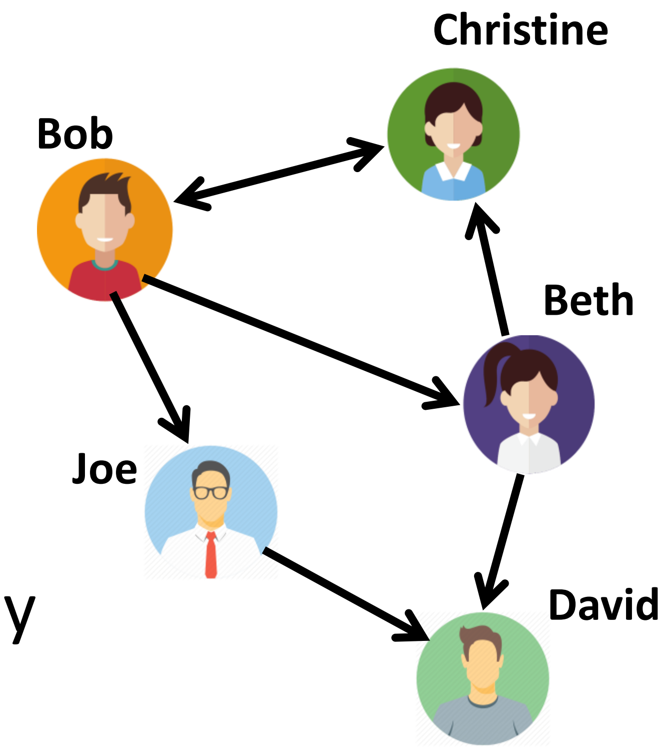
Propagation over networks with variable influence



[Gomez-Rodriguez et al., WSDM 2013]

Recurrent events: beyond cascades

Up to this point, each users is only infected once, and event sequences can be seen as cascades.



In general, users perform recurrent events over time. E.g., people repeatedly express their opinion online:

CNN
How social media is revolutionizing debates

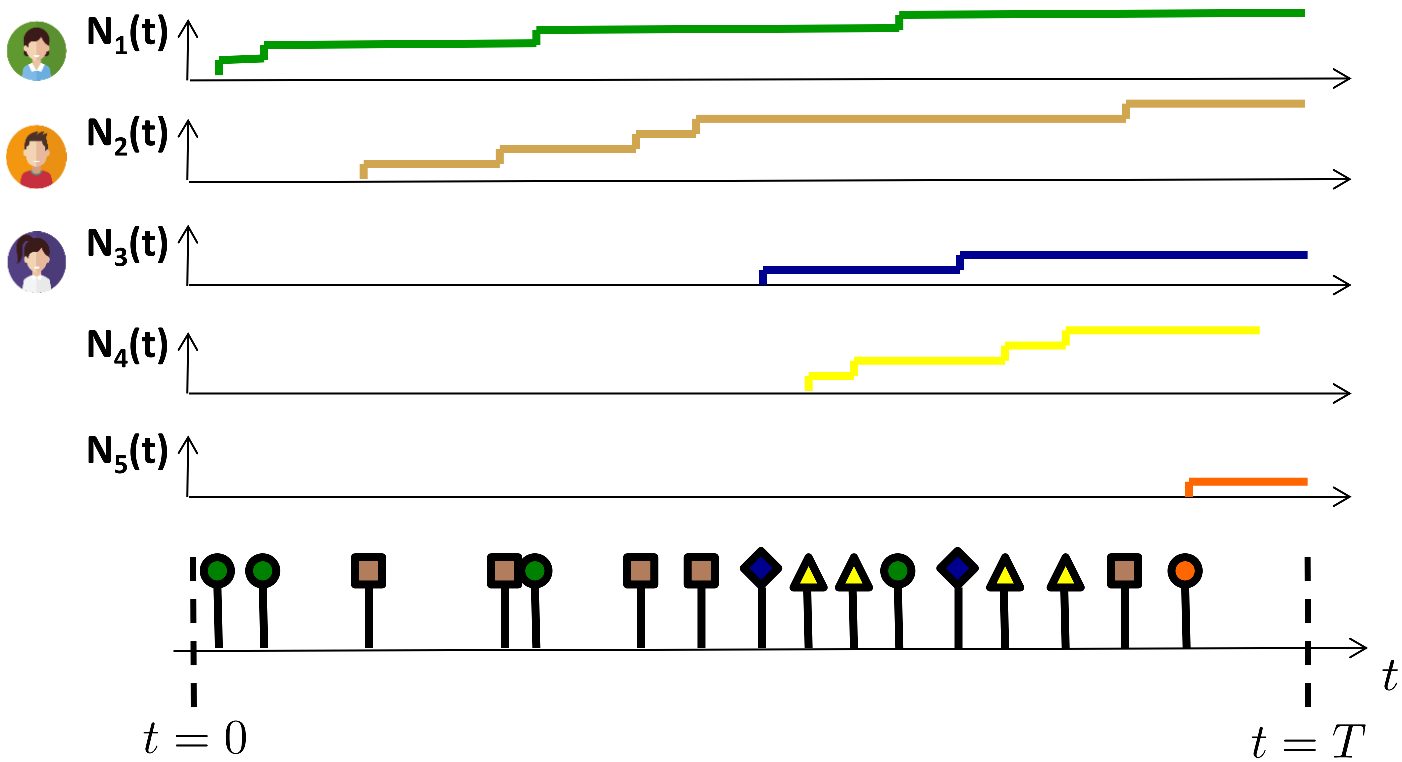
The New York Times
Social Media Are Giving a Voice to Taste Buds

TC
Twitter Unveils A New Set Of Brand-Centric Analytics

The New York Times
Campaigns Use Social Media to Lure Younger Voters

Recurrent events representation

We represent messages using **nonterminating temporal point processes**:



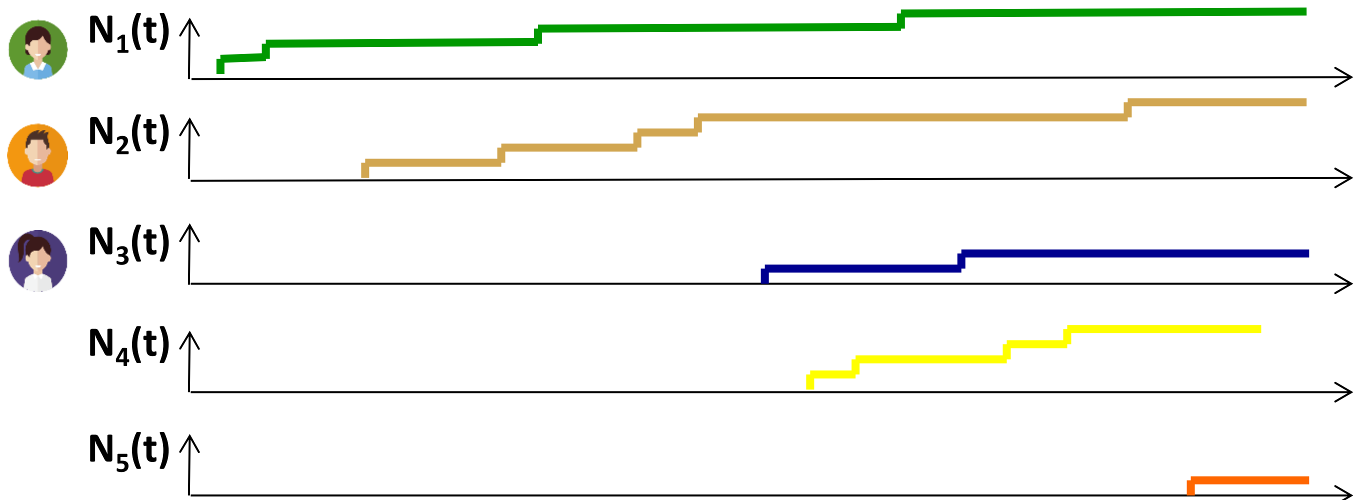
Recurrent event:

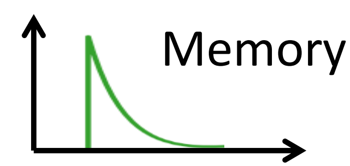
$$(u_i, t_i)$$

User **Time**

[Farajtabar et al., NIPS 2014]

Recurrent events intensity





Cascade sources!

$$\underbrace{\lambda_u^*(t)}_{\text{User's intensity}} = \underbrace{\mu_u}_{\text{Events on her own initiative}} + \sum_{v\in[m]} b_{vu} \underbrace{\sum_{e_i\in\mathcal{H}_v(t)} \kappa(t-t_i)}_{\text{Previous messages by user v}}$$

Influence from user v on user u

Hawkes process

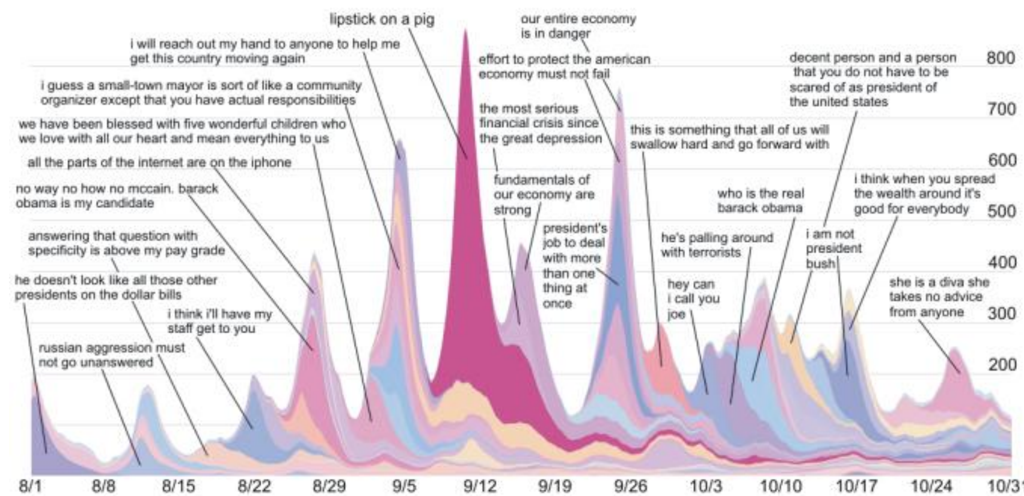
[De et al., NIPS 2016]

Models & Inference

1. Modeling event sequences
2. **Clustering event sequences**
3. Capturing complex dynamics
4. Causal reasoning on event sequences

Event sequences

So far, we have assumed the cascade (topic, meme, etc.) that each event belongs to was known.



Often, the cluster (topic, meme, etc.) that each event in a sequence belongs to is not known:

BBC News (World) @BBCWorld · 4m
Turkey election: Erdogan win ushers in new presidential era

BBC News (World) @BBCWorld · 46m
Dublin church: Seven injured as car hits pedestrians

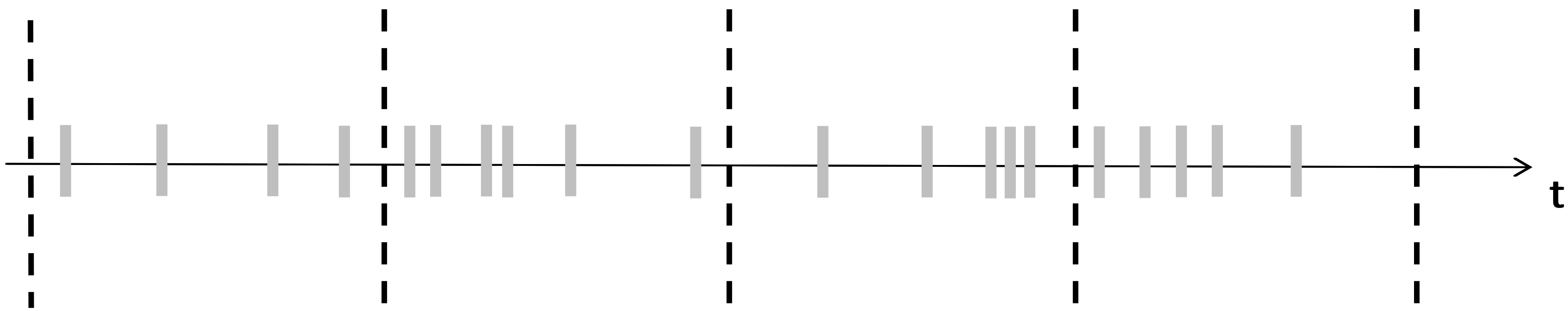
BBC News (World) @BBCWorld · 2h
Nigerian music star D'banj's son 'drowns at home'

BBC News (World) @BBCWorld · 2h
Turkey election: Country's heart split over Erdogan victory

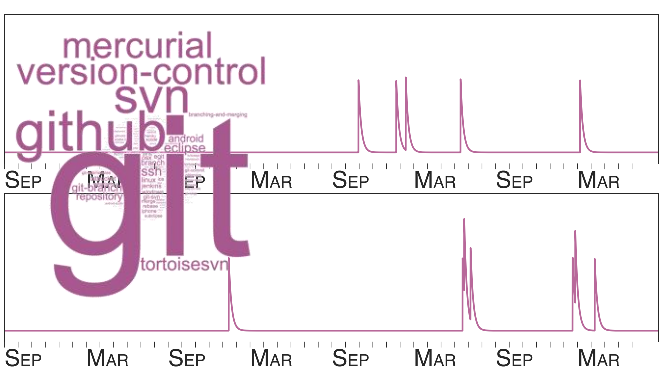
Politics

Music

Assume the event **cluster to be hidden** and aim to automatically **learn the cluster assigments** from the data:

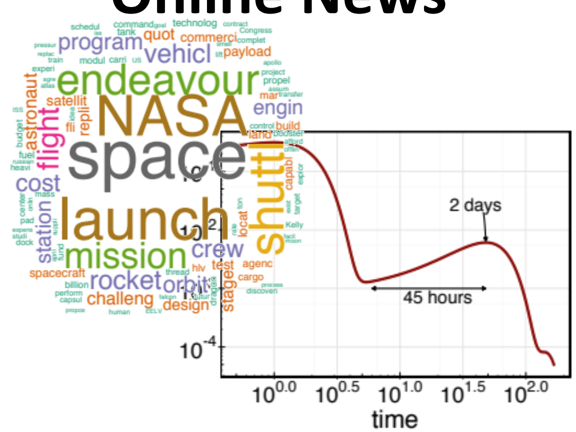


Bayesian methods to cluster event sequences in the context of:



Learning

Online News



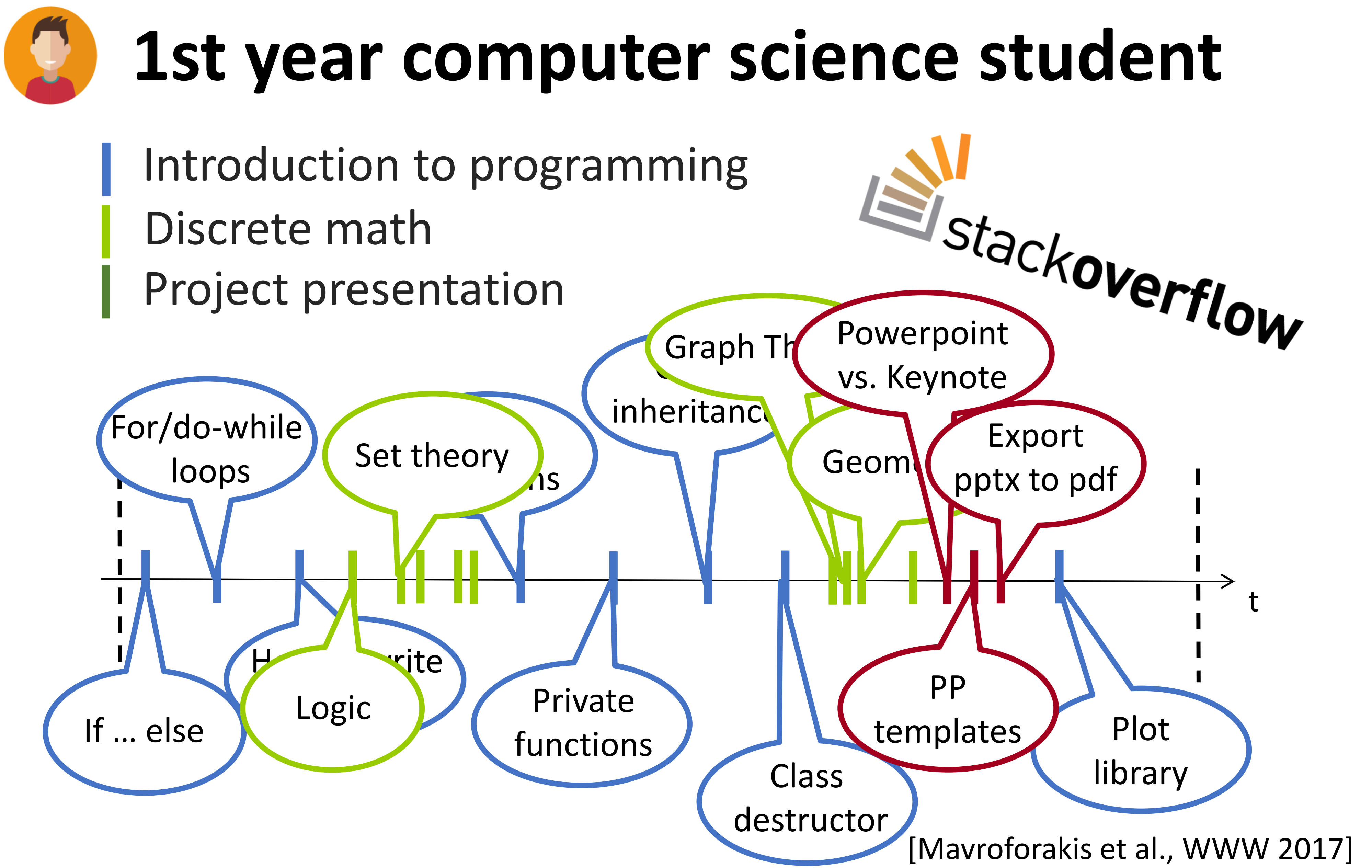
Health care

Method	DMHP
ICU Patient	**0.3778**
IPTV User	**0.2004**

[Du et al., 2015; Mavroforakis et al., 2017; Xu & Zha, 2017]

Hierarchical Dirichlet Hawkes process

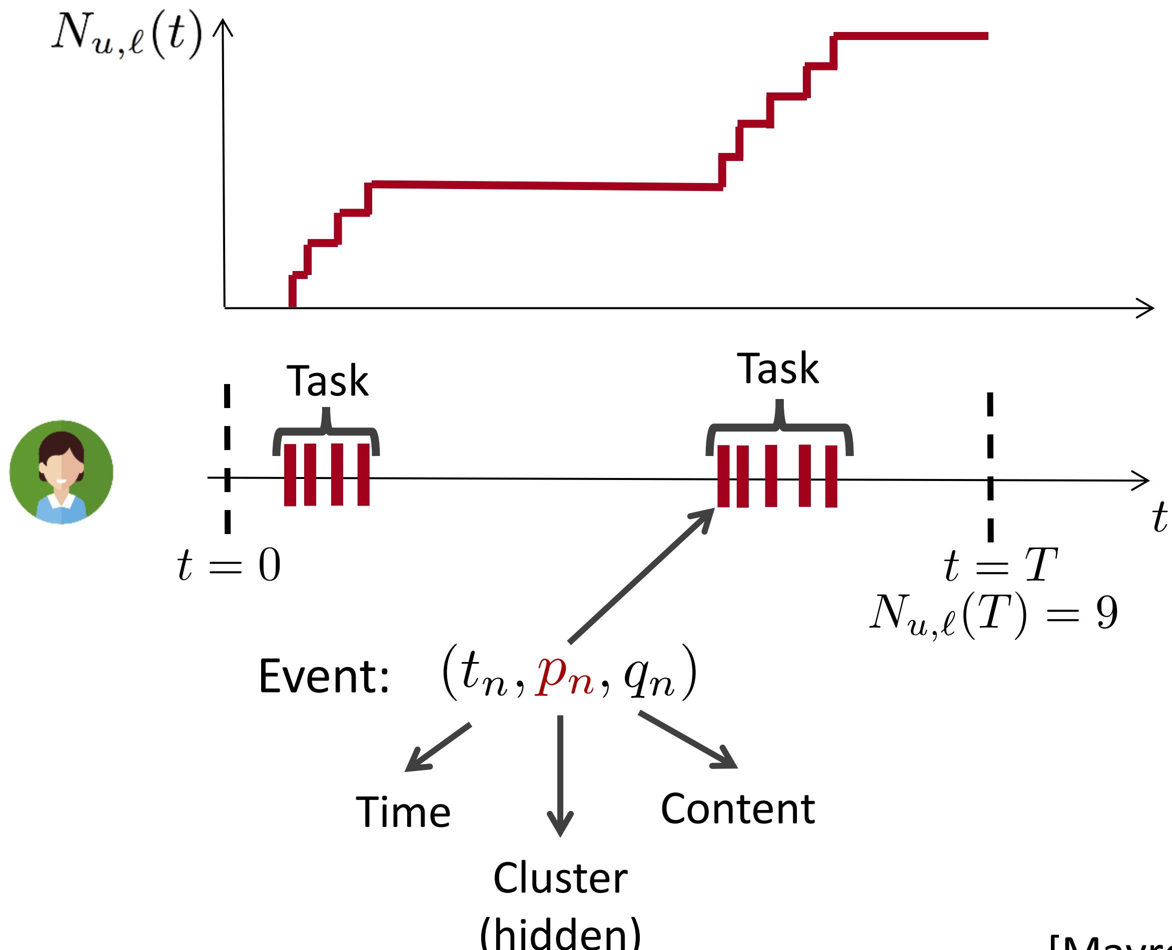
1st year computer science student



[Mavroforakis et al., WWW 2017]

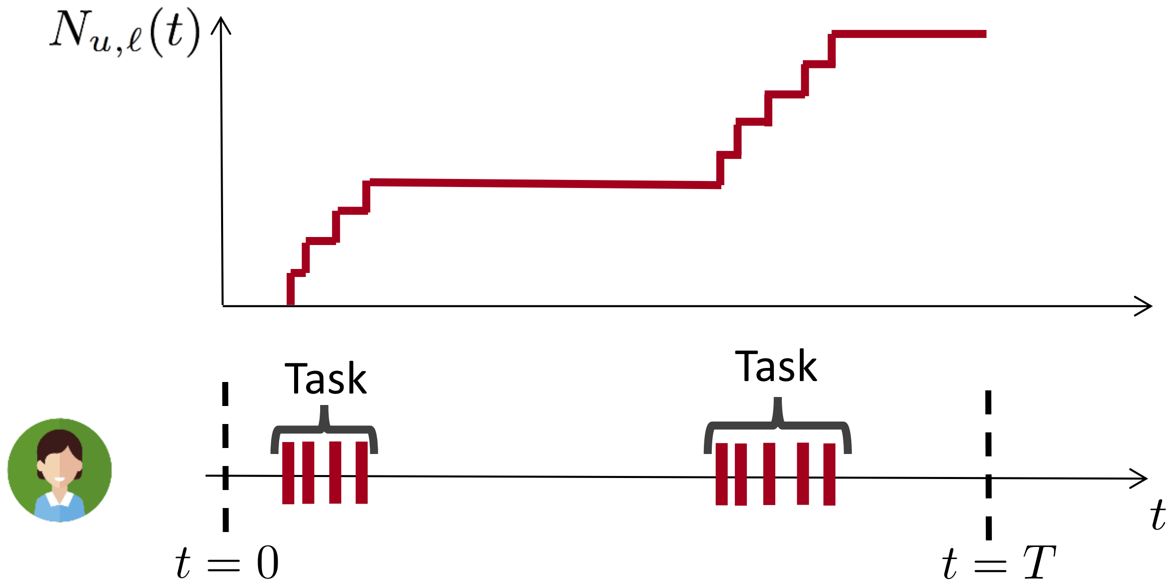
Events representation

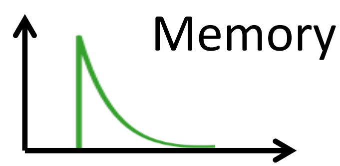
We represent the events using **marked temporal point processes**:



[Mavroforakis et al., WWW 2017]

Cluster intensity





New cascade rate

Cluster popularity

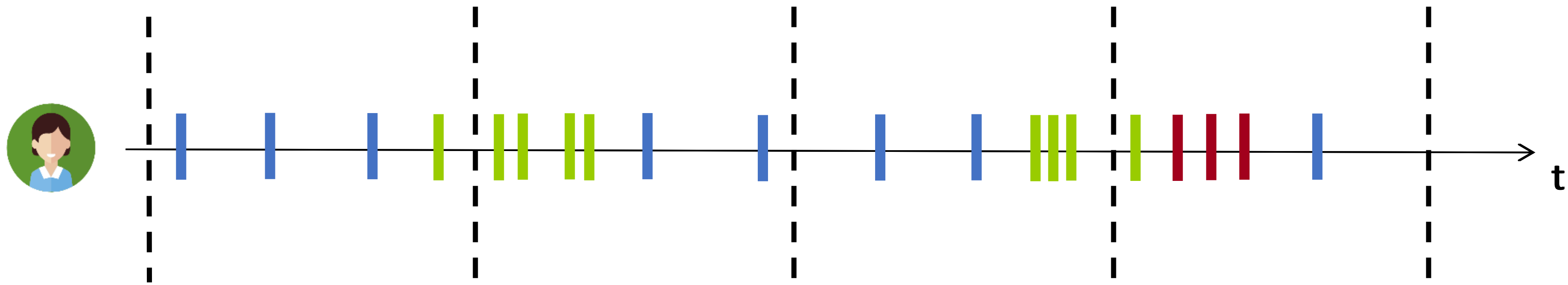
$$\underbrace{\lambda^*_{u,\ell}(t)}_{\substack{\textbf{Intensity} \\ \textbf{or rate} \\ \textbf{(events / hour)}}} = \underbrace{\mu_u \pi_\ell}_{\substack{\textbf{Own} \\ \textbf{initiative}}} + \underbrace{\sum_{j:t_j \in \mathcal{H}_{u,\ell}(t)} k_{\theta_\ell}(t - t_j)}_{\textbf{Follow-up}}$$

Hawkes process

[Mavroforakis et al., WWW 2017]

User events intensity

Users adopt more than one cluster:



A user's learning events as a multidimensional Hawkes:

Time, cluster

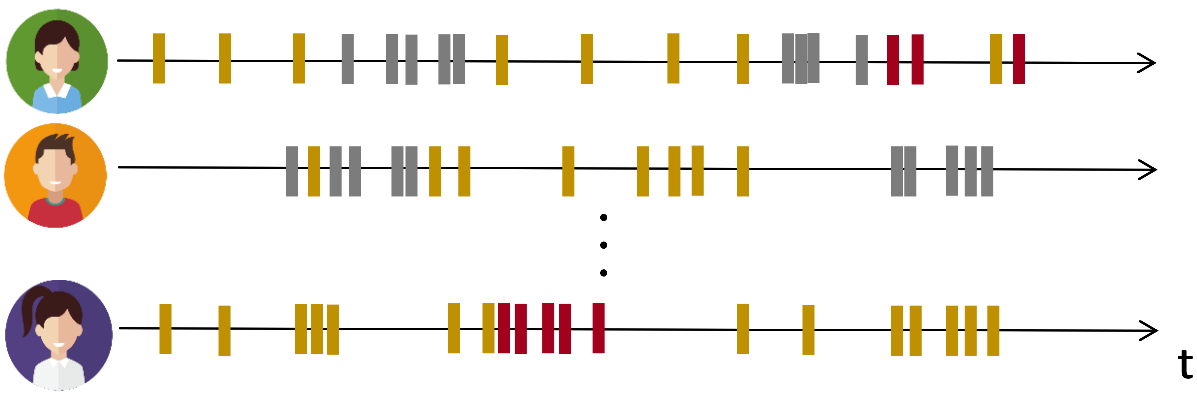
$$(t_n, p_n) \sim Hawkes \begin{pmatrix} \lambda^*_{u,1}(t) \\ \vdots \\ \lambda^*_{u,\infty}(t) \end{pmatrix}$$

Content $\rightarrow$ $q_n = \boldsymbol{\omega}$ $\quad \omega_j \sim Multinomial(\boldsymbol{\theta}_p)$

[Mavroforakis et al., WWW 2017]

People share same clusters

Different users adopt same clusters



Cluster distribution from a **Dirichlet process:**

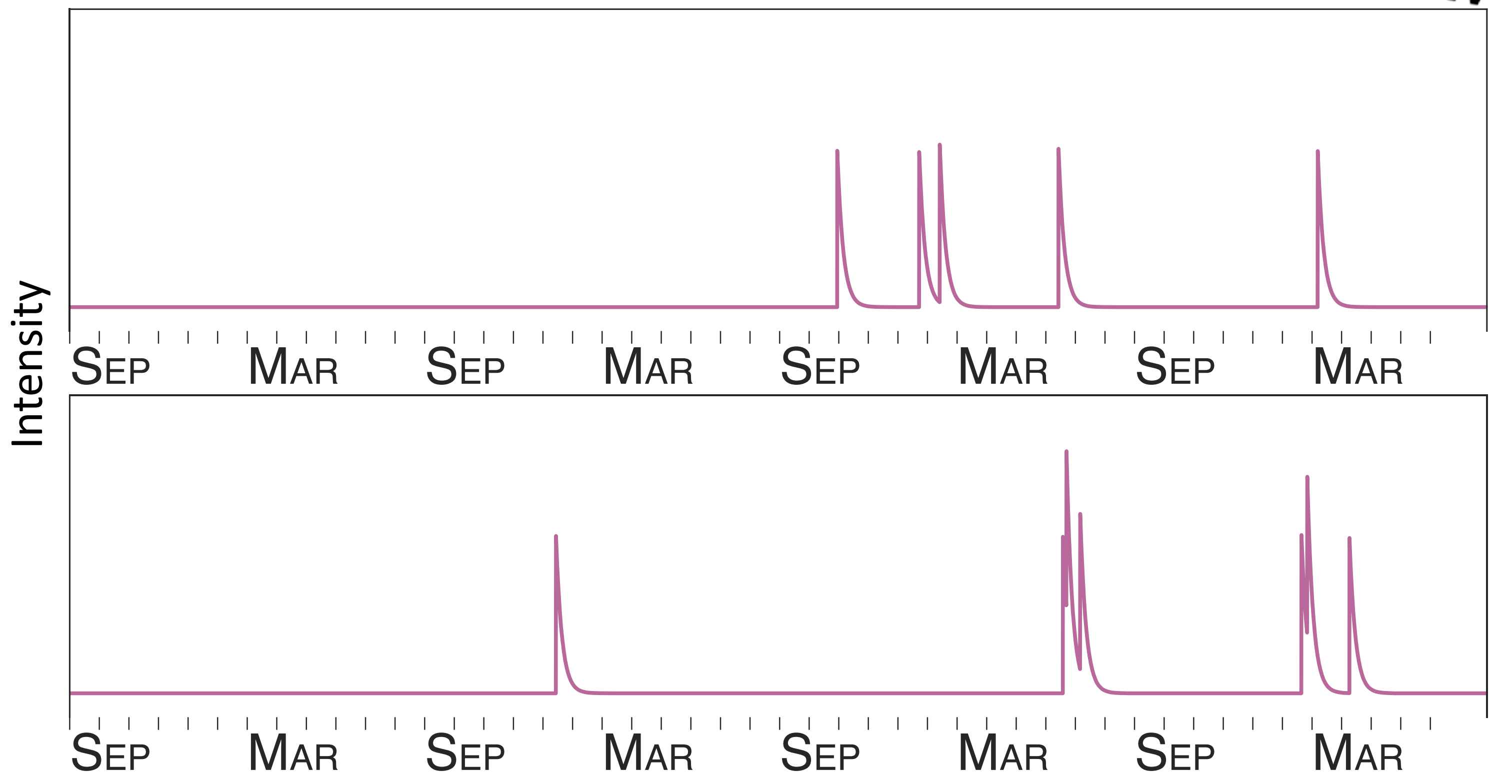
- Infinite # of clusters.
- Shared parameters across users.

Details in the reference below!

[Mavroforakis et al., WWW 2017]

Content

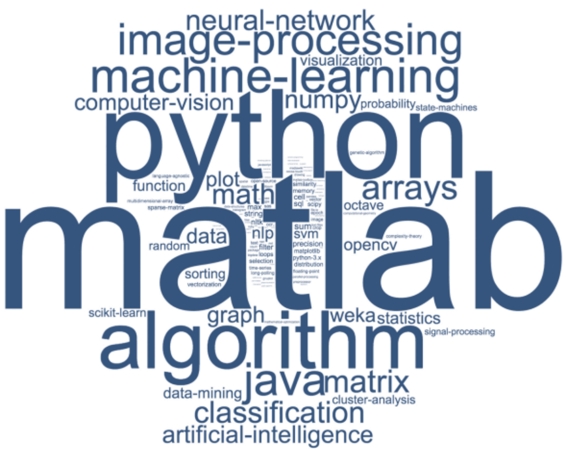
Intensities



Version control tasks tend to be specific, quickly solved after performing few questions

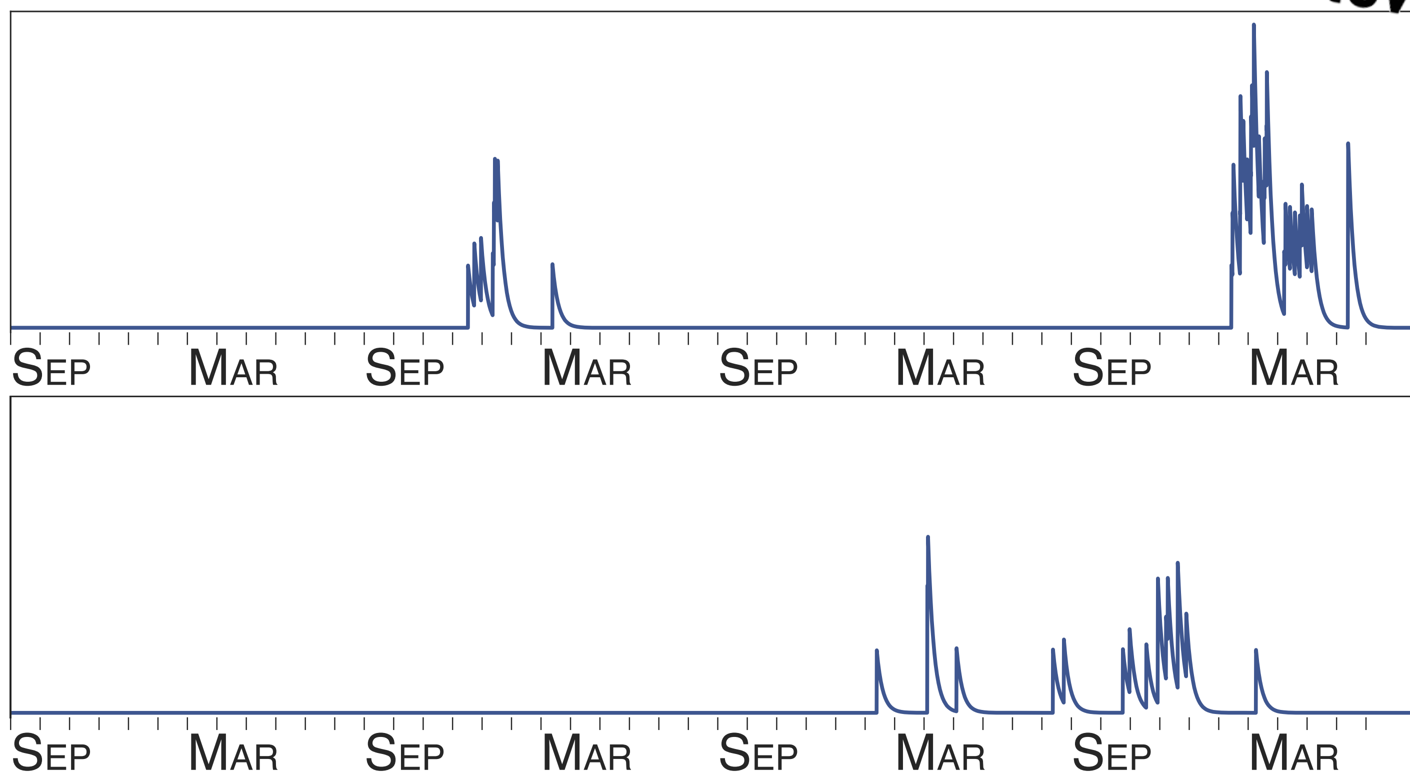
[Mavroforakis et al., WWW 2017]

Content



Intensities

stackoverflow



Machine learning tasks tend to be more complex and require asking more questions

[Mavroforakis et al., WWW 2017]

Models & Inference

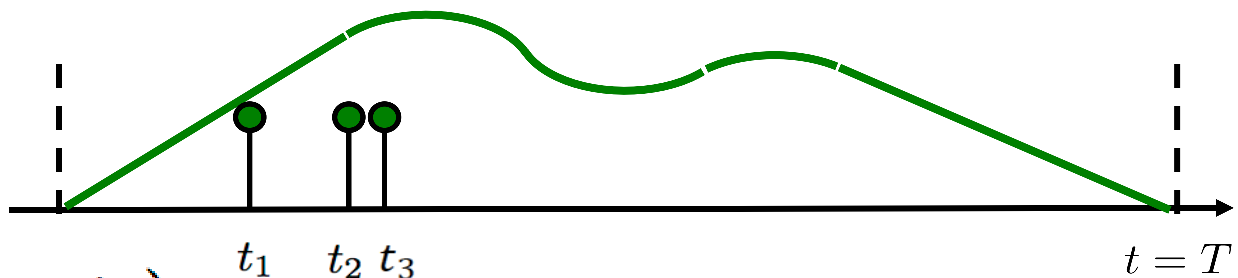
1. Modeling event sequences
2. Clustering event sequences
3. **Capturing complex dynamics**
4. Causal reasoning on event sequences

Case Studies & References

For those who want to do research in social media

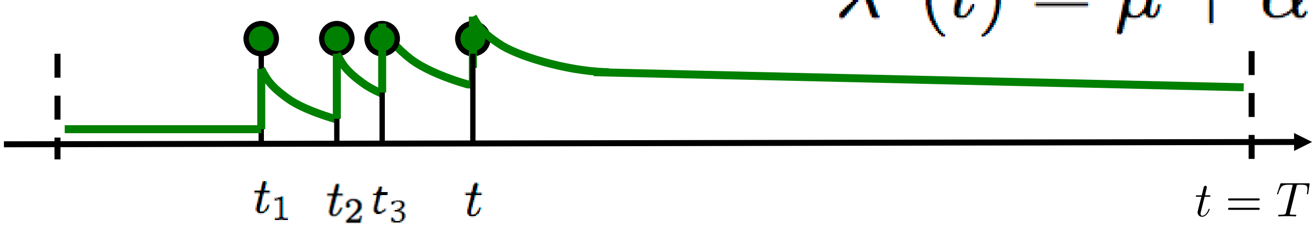
Up to now, we have focused on simple temporal dynamics (and intensity functions):

$$\lambda^*(t) = \mu$$



$$\lambda^*(t) = \sum_j \alpha_j \, k(t - t_j)$$

$$\lambda^*(t) = \mu + \alpha \sum_{t_i \in \mathcal{H}(t)} \kappa_\omega(t - t_i)$$



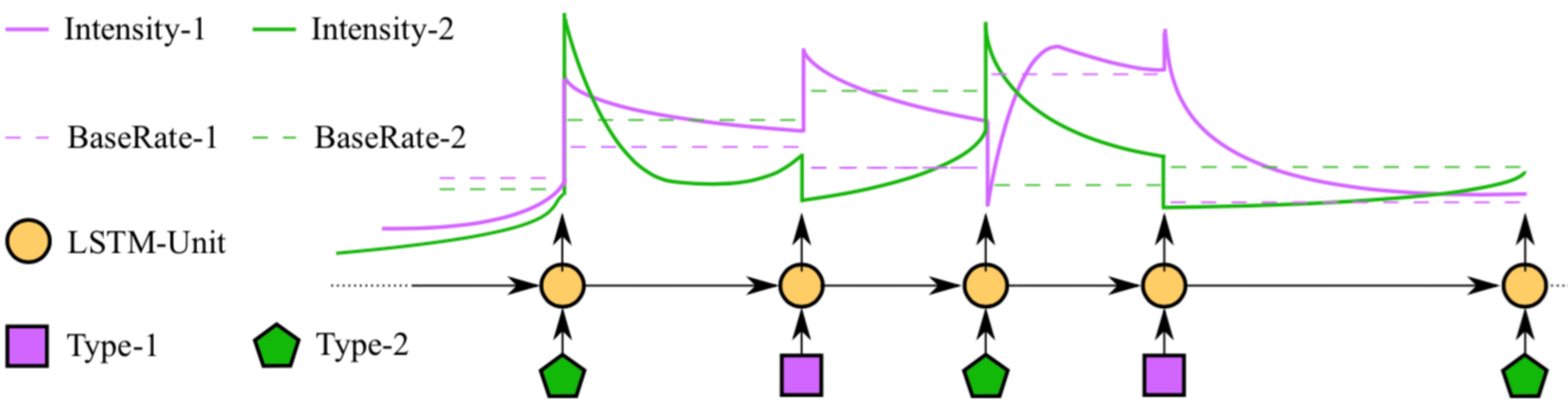
Recent works make use of RNNs to capture more complex dynamics

[Du et al., 2016; Dai et al., 2016; Mei & Eisner, 2017; Jing & Smola, 2017; Trivedi et al., 2017; Xiao et al., 2017a; 2018]

Neural Hawkes process

1) History effect does not need to be additive

2) Allows for complex memory effects
   (such as delays)



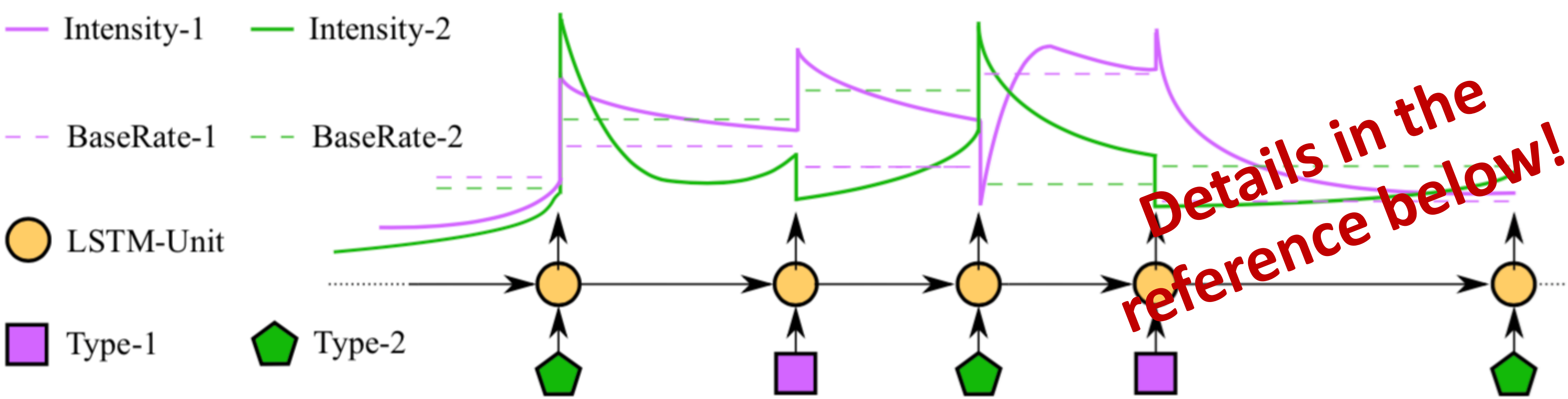
[Mei & Eisner, NIPS 2017]

Neural Hawkes process

$$\lambda_u(t) = f_u(\mathbf{w}_u^\top \mathbf{h}(t))$$

Excitation & inhibition

$$\mathbf{h}(t) = \overbrace{\mathrm{RNN}(\mathcal{H}(t))}^{\textbf{Memory}}$$

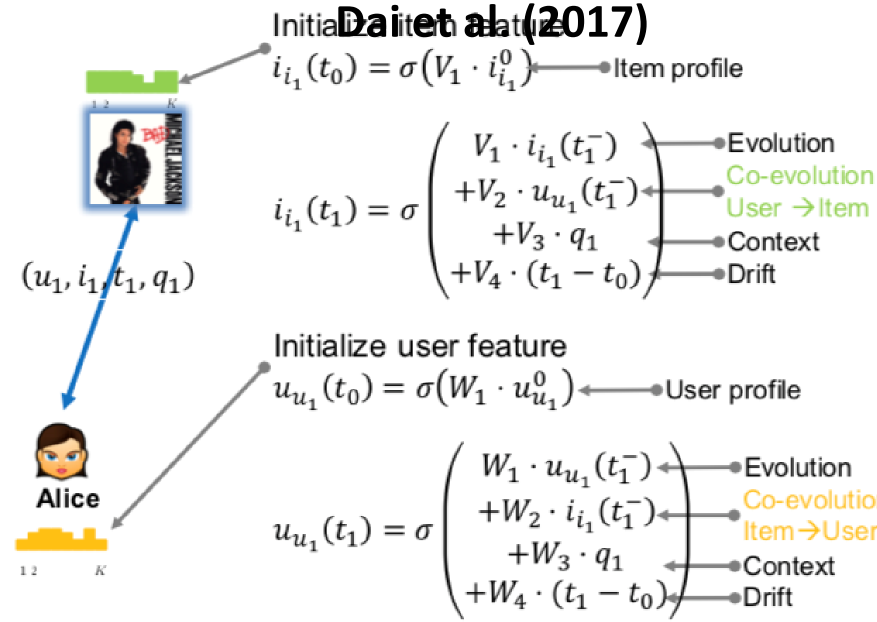


[Mei & Eisner, NIPS 2017]

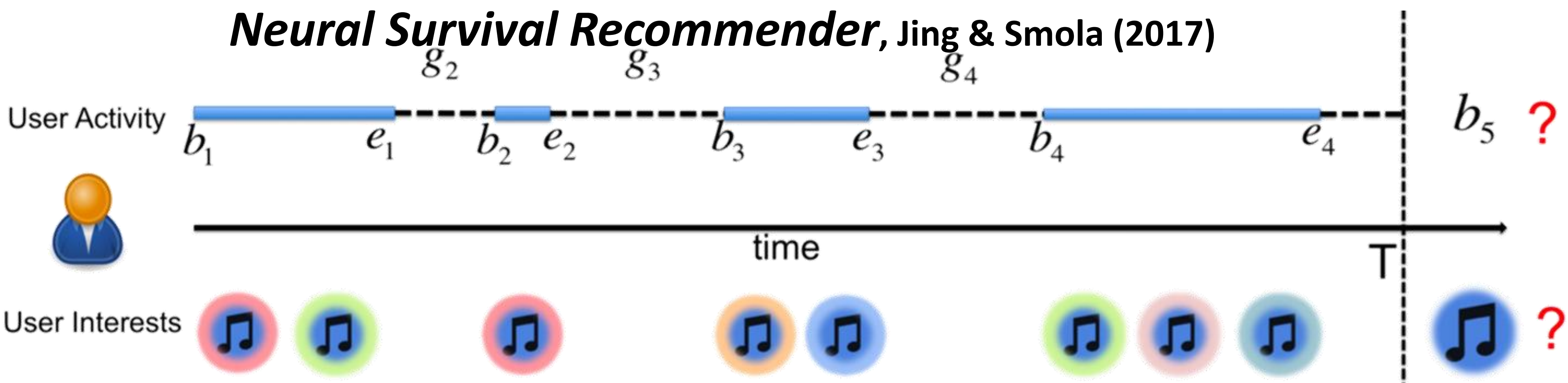
Applications (I): Predictive Models

Know-Evolve, **Trivedi et al. (2017)**

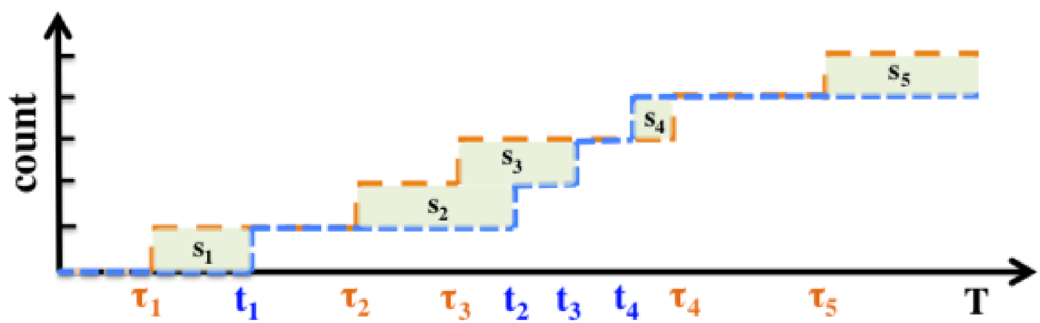
Coevolutionary Embedding, **Dai et al. (2017)**



Neural Survival Recommender, **Jing & Smola (2017)**

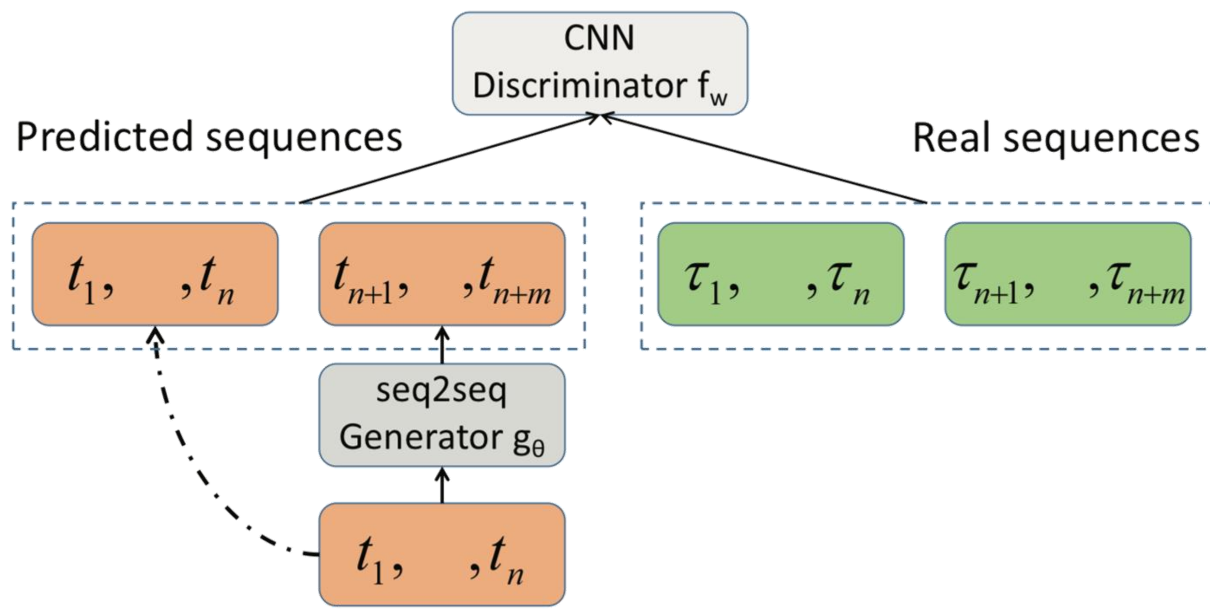


Key idea: Intensity- and likelihood-free models



Wasserstein-Distance for Temporal Point Processes

GAN architecture



[Xiao et al., 2017 & 2018]

Models & Inference

1. Modeling event sequences
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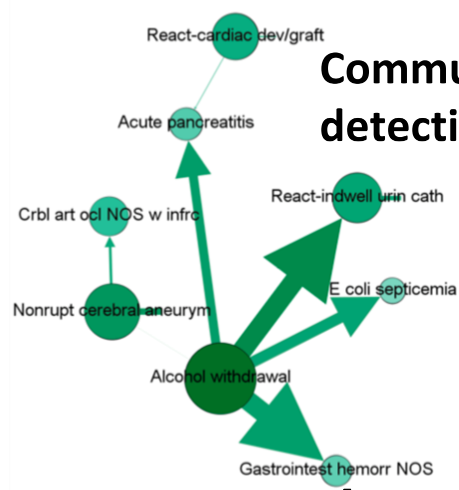
Temporal point processes beyond prediction

So far, we have focused on models that improve preditions:

Link prediction

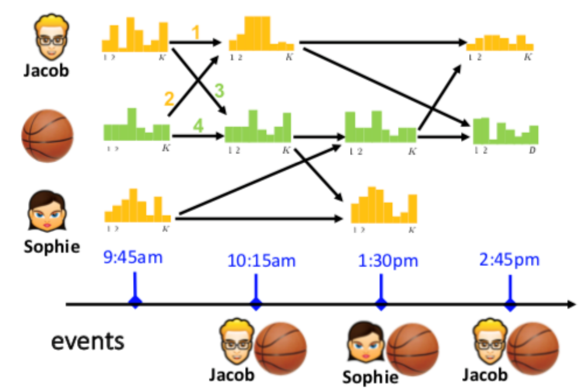
[Trivedi et al., 2017]

Community detection



[Xiao et al., 2017]

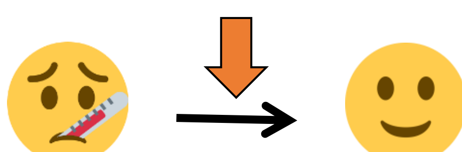
Recommendations



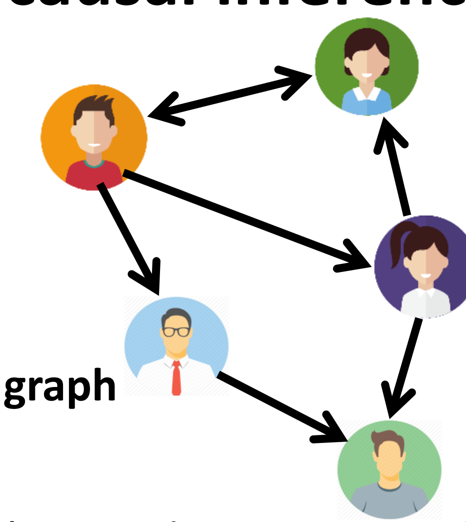
[Dai et al., 2017]

Recent works have focused on performing **causal inference using event sequences**:

Treatment effect



Granger causality graph



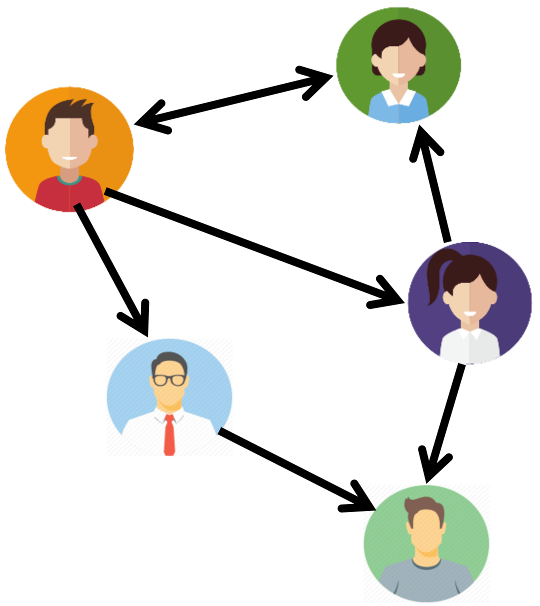
[Xu et al., 2016; Achab et al., 2017; Kuśmierczyk & Gomez-Rodriguez, 2018]

Uncovering Causality from Hawkes Processes

Multivariate Hawkes process:

$$N(t) = \sum_{u \in \mathcal{U}} N_u(t)$$

$$\lambda_u(t) = \mu_u + \sum_{v \in \mathcal{U}} \int_0^t \underbrace{k_{u,v}(t-t')}_{\textbf{Effect of v's past events on u}} dN_v(t')$$



Granger causality:

"X causes Y in the sense of Granger causality if forecasting future values of Y is more successful while taking X past values into account" [Granger, 1969]

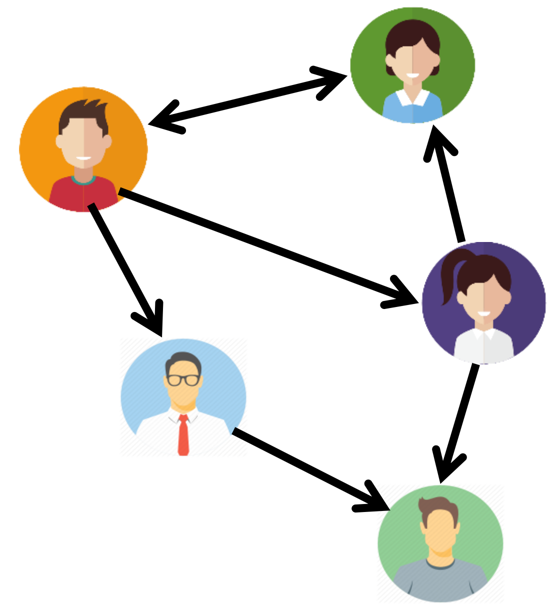
[Achab et al., ICML 2017]

Uncovering Causality from Hawkes Processes

Multivariate Hawkes process:

$$N(t) = \sum_{u \in \mathcal{U}} N_u(t)$$

$$\lambda_u(t) = \mu_u + \sum_{v \in \mathcal{U}} \int_0^t \underbrace{k_{u,v}(t - t')}_{\text{Effect of v's past events on u}} dN_v(t')$$



Granger causality on multivariate Hawkes processes:

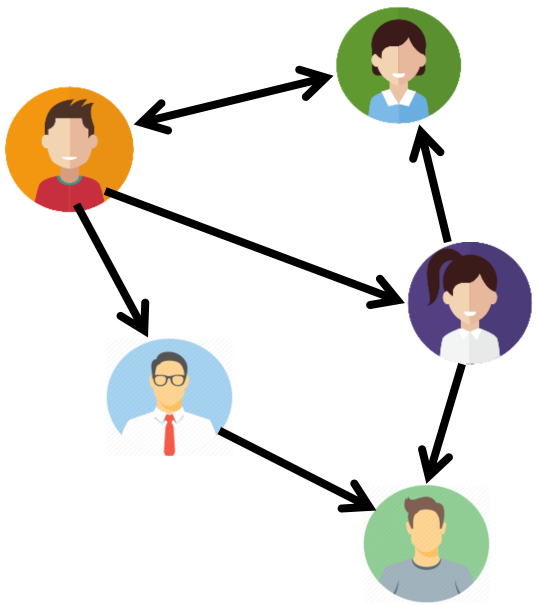
“ $N_v(t)$ does not Ganger-cause $N_u(t)$ w.r.t. $N(t)$ if and only if $k_{u,v}(\tau) = 0$ for $\tau \in \Re^+$ "

[Eichler et al., 2016]

[Achab et al., ICML 2017]

Uncovering Causality from Hawkes Processes

Goal is to estimate $G = [g_{uv}]$, where:



$$g_{uv} = \int_0^{+\infty} k_{u,v}(\tau) d\tau \geq 0 \text{ for all } u, v \in \mathcal{U}$$

Average total # of events of node u whose *direct* ancestor is an event by node v

Then, $G = [g_{uv}]$ quantifies the *direct causal relationship* between nodes.

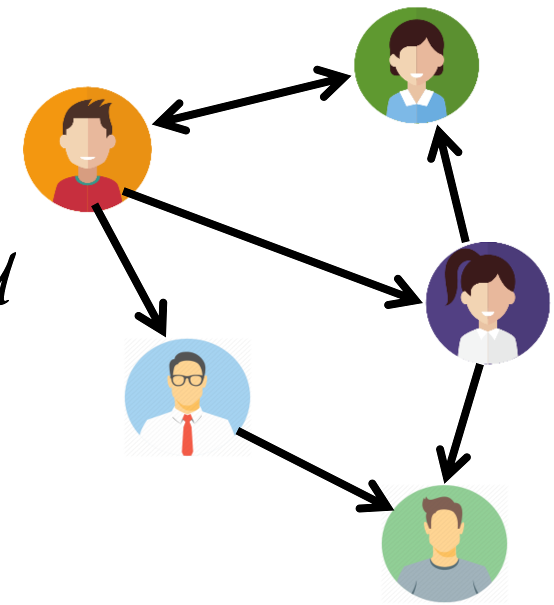
[Achab et al., ICML 2017]

Uncovering Causality from Hawkes Processes

Goal is to estimate $G = [g_{uv}]$**, where:**

$$g_{uv} = \int_0^{+\infty} k_{u,v}(\tau)d\tau \geq 0 \text{ for all } u, v \in \mathcal{U}$$

Average total # of events of node u whose *direct* ancestor is an event by node v



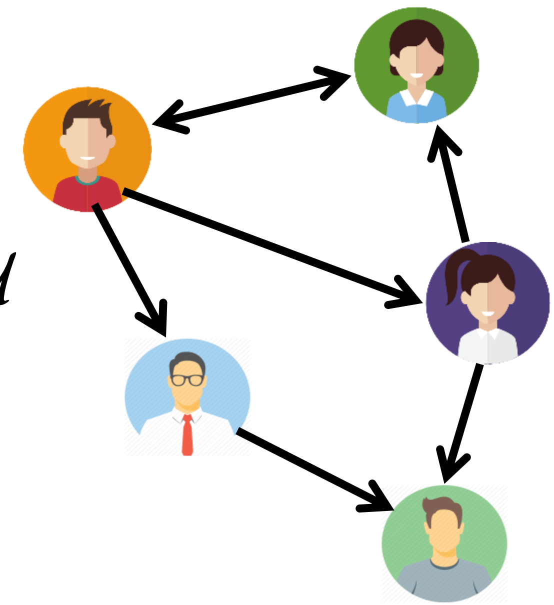
Then, $G = [g_{uv}]$ **quantifies the *direct causal relationship* between nodes.**

Key idea: Estimate G using the cumulants dN(t) of the Hawkes process.

[Achab et al., ICML 2017]

Uncovering Causality from Hawkes Processes

Goal is to estimate $G = [g_{uv}]$, where:



$$g_{uv} = \int_0^{+\infty} k_{u,v}(\tau) d\tau \geq 0 \text{ for all } u, v \in \mathcal{U}$$

Average total # of events of node u whose *direct* ancestor is an event by node v

Then, $G = [g_{uv}]$ quantifies the *direct causal relationship* between nodes.

Details in the reference below!

Key idea: Estimate G using the cumulants the dN(t) of the Hawkes process.

[Achab et al., ICML 2017]

Next Week:

Gaussian Process

Have a good day!