In the name of GOD.

Sharif University of Technology

Stochastic Processes

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Quiz 7 (25 minutes) | Estimation Theory 2 | Date: Dec. 23th

1. **Question 1.**

   Let $Y \in \{0,1\}$ satisfy

   $$\mathbb{P}(Y=0)=\alpha, \qquad \mathbb{P}(Y=1)=1-\alpha, \quad \alpha \in (0,1) \text{ known.}$$

   Assume μ is known and

   $$X \mid Y=0 \sim \mathcal{N}(0,\sigma^2), \qquad X \mid Y=1 \sim \mathcal{N}(\mu,\sigma^2).$$

   - Suppose (X_i, Y_i), $i=1,\ldots,n$, are i.i.d. Write the likelihood and derive the MLE of σ^2.

2. **Question 2.**

   Let $X_1,\ldots,X_n$ be i.i.d. Gamma(α,β) with shape $\alpha>0$ and scale $\beta>0$. You are given

   $$\mathbb{E}[X]=\alpha\beta, \qquad \mathrm{Var}(X)=\alpha\beta^2.$$

   Define the sample raw moments

   $$m_1=\frac{1}{n}\sum_{i=1}^{n} X_i, \qquad m_2=\frac{1}{n}\sum_{i=1}^{n} X_i^2.$$

   Using the method of moments based on m_1, m_2, derive estimators $\hat{\alpha}$ and $\hat{\beta}$ in terms of m_1 and m_2.

3. **Question 3.**

   Suppose $X_1,\ldots,X_n$ are i.i.d. observations from a parametric family, and let $A(X)$ and $B(X)$ be two estimators of a parameter θ. Assume that

   $$\mathrm{Var}_\theta(A) < \mathrm{Var}_\theta(B) \quad \text{for all } \theta.$$

   Does this imply that $A(X)$ is preferred in the sense of UMVUE or that it attains the Cramér–Rao lower bound? Explain.

4. **Solution 1.**

   Let

   $$S_0=\{i : Y_i=0\}, \quad S_1=\{i : Y_i=1\}, \quad n_0=|S_0|, \quad n_1=|S_1|, \quad n=n_0+n_1.$$

Likelihood (up to constant factors):

$$L(\sigma^2) = \prod_{i\in S_0} \frac{1}{\sqrt{2\pi}\sigma} e^{-X_i^2/(2\sigma^2)} \prod_{i\in S_1} \frac{1}{\sqrt{2\pi}\sigma} e^{-(X_i-\mu)^2/(2\sigma^2)}.$$

Log-likelihood (up to constants):

$$\ell(\sigma^2) = -\frac{n}{2}\log(\sigma^2) - \frac{1}{2\sigma^2}\Big(\sum_{i\in S_0} X_i^2 + \sum_{i\in S_1}(X_i-\mu)^2\Big).$$

Let

$$A = \sum_{i\in S_0} X_i^2 + \sum_{i\in S_1}(X_i-\mu)^2.$$

Derivative:

$$\frac{\partial \ell}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \frac{A}{2(\sigma^2)^2}.$$

Setting equal to zero gives

$$\hat{\sigma}^2 = \frac{A}{n} = \frac{1}{n}\left(\sum_{i\in S_0} X_i^2 + \sum_{i\in S_1}(X_i-\mu)^2\right).$$

Second derivative:

$$\frac{\partial^2 \ell}{\partial(\sigma^2)^2} = \frac{n}{2(\sigma^2)^2} - \frac{A}{(\sigma^2)^3} = -\frac{n}{2(\sigma^2)^2} < 0,$$

so the solution maximizes $\ell(\sigma^2)$.

5. **Solution 2.**

   From $\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$, we have

   $$\mathbb{E}[X^2] = \mathrm{Var}(X) + (\mathbb{E}[X])^2 = \alpha\beta^2 + (\alpha\beta)^2 = \alpha(\alpha+1)\beta^2.$$

   Method of moments matches raw moments:

   $$m_1 = \mathbb{E}[X] = \alpha\beta, \qquad m_2 = \mathbb{E}[X^2] = \alpha(\alpha+1)\beta^2.$$

   From $m_1 = \alpha\beta$, $\beta = \frac{m_1}{\alpha}$. Substitute into the second equation:

   $$m_2 = \alpha(\alpha+1)\left(\frac{m_1}{\alpha}\right)^2 = \frac{\alpha+1}{\alpha} m_1^2.$$

   Thus

   $$\frac{m_2}{m_1^2} = 1 + \frac{1}{\alpha} \quad \Longrightarrow \quad \boxed{\hat{\alpha} = \frac{m_1^2}{m_2 - m_1^2}}.$$

   Finally,

   $$\boxed{\hat{\beta} = \frac{m_1}{\hat{\alpha}} = \frac{m_2 - m_1^2}{m_1}}.$$

6. **Solution 3.**

   Even if an estimator $A(X)$ has strictly smaller variance than another estimator $B(X)$ for all parameter values, it is not automatically preferred in the context of UMVUE or the Cramér–Rao lower bound. This is because both UMVUE and CRLB are defined only among unbiased estimators. If $A(X)$ is biased, it is excluded from consideration regardless of its variance. Therefore, variance alone does not determine optimality; unbiasedness is a necessary condition.