In the name of GOD.

Sharif University of Technology

Stochastic Processes

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Quiz 3 (20 minutes)	Ergodicity + Power Spectrum	Date: Aban 18th

1. A scientist measures a random process $X(t)$ and records a single, infinitely long sample path. The observed signal appears as pure noise with a DC offset (a constant value that is added to the "pure noise" signal). This DC offset was determined by a single random coin toss before the experiment, equal to $+10$ or -10 with equal probability. Each realization of the process has its own constant offset determined the same way.

   The scientist claims that this process is ergodic in the mean. Discuss the validity of this claim.

   Solution: The claim is false.

   - The ensemble average is $E[X(t)] = E[\text{DC offset}] = 0$.
   - The time average for a single realization is equal to its specific DC offset, either $+10$ or -10.
   - Since the time average of one realization depends on a random outcome and does not equal the ensemble mean, the process is not ergodic in the mean.

   This is analogous to the example $X(t) = C$ in slides, where C is a random variable.

2. A Wide-Sense Stationary (WSS) stochastic process $X(t)$ is passed through a Linear Time-Invariant (LTI) system with an impulse response of $h(t) = 5\delta(t-2)$, where $\delta(t)$ is the Dirac delta function.

   The input process $X(t)$ has an autocorrelation function given by

$$R_{XX}(\tau) = 4e^{-3|\tau|}$$

   Calculate the Power Spectral Density of the output process $Y(t)$, denoted as $S_{YY}(\omega)$.

Time Domain ($g(t)$)	**Frequency Domain ($G(\omega)$)**	**Notes**
$\delta(t)$	1	Dirac Delta
$\delta(t-t_0)$	$e^{-j\omega t_0}$	Time-Shift Property
$e^{-a\|t\|}$ (for $a>0$)	$\frac{2a}{a^2+\omega^2}$	Two-Sided Exponential
$u(t)$	$\pi\delta(\omega)+\frac{1}{j\omega}$	Unit Step Function
$\Pi\left(\frac{t}{\tau}\right) = \begin{cases} 1 & \|t\| \le \tau/2 \\ 0 & \|t\| > \tau/2 \end{cases}$	$\tau\,\text{sinc}\left(\frac{\omega\tau}{2\pi}\right)$	Rectangular Pulse
$e^{-at}u(t)$ (for $a>0$)	$\frac{1}{a+j\omega}$	Causal Exponential

Solution:

The output Power Spectral Density $S_{YY}(\omega)$ is related to the input Power Spectral Density $S_{XX}(\omega)$ and the system's transfer function $H(\omega)$ by the formula:

$$S_{YY}(\omega) = S_{XX}(\omega)|H(\omega)|^2$$

Step 1: Find the Input Power Spectral Density, $S_{XX}(\omega)$

The input PSD is the Fourier Transform of the input autocorrelation function $R_{XX}(\tau)$ (Wiener-Khinchin Theorem).

Given $R_{XX}(\tau) = 4e^{-3|\tau|}$.

We use the Fourier Transform pair: $\mathcal{F}\{e^{-a|\tau|}\} = \frac{2a}{a^2+\omega^2}$. With $a = 3$:

$$S_{XX}(\omega) = \mathcal{F}\{R_{XX}(\tau)\} = \mathcal{F}\{4e^{-3|\tau|}\} = 4 \cdot \frac{2(3)}{3^2 + \omega^2} = \frac{24}{9 + \omega^2}$$

Step 2: Find the System Transfer Function, $H(\omega)$

The transfer function $H(\omega)$ is the Fourier Transform of the impulse response $h(t)$.

Given $h(t) = 5\delta(t - 2)$.

We use the Fourier Transform property of time shift: $\mathcal{F}\{\delta(t - t_0)\} = e^{-j\omega t_0}$. With $t_0 = 2$:

$$H(\omega) = \mathcal{F}\{h(t)\} = \mathcal{F}\{5\delta(t - 2)\} = 5e^{-j2\omega}$$

Step 3: Calculate the Squared Magnitude of the Transfer Function, $|H(\omega)|^2$

$$|H(\omega)|^2 = |5e^{-j2\omega}|^2 = |5|^2 \cdot |e^{-j2\omega}|^2$$

Since $|e^{-j\theta}| = 1$ for any real θ:

$$|H(\omega)|^2 = 25 \cdot 1 = 25$$

Step 4: Calculate the Output Power Spectral Density, $S_{YY}(\omega)$

Substitute $S_{XX}(\omega)$ and $|H(\omega)|^2$ into the main equation:

$$S_{YY}(\omega) = S_{XX}(\omega)|H(\omega)|^2$$

$$S_{YY}(\omega) = \left(\frac{24}{9 + \omega^2}\right) \cdot 25$$

$$S_{YY}(\omega) = \frac{600}{9 + \omega^2}$$

Final Answer

The Power Spectral Density of the output process $Y(t)$ is:

$$\mathbf{S_{YY}}(\omega) = \frac{\mathbf{600}}{\mathbf{9} + \omega^{\mathbf{2}}}$$