In the name of GOD.

Stochastic Processes

Fall 2025

Hamid R. Rabiee

Sharif University of Technology

Quiz 1 (15 minutes)	Probability Concepts - Review	Date: Oct. 21th

1. (a) Define **independence** and **uncorrelatedness** for two random variables, X and Y, each in a single sentence, and provide their corresponding mathematical formulas (**15 pts**).

   (b) Let X be a random variable with a uniform distribution, $X \sim U(a, b)$, and let $Y = X^2$. Are X and Y independent? Under what condition will X and Y be **uncorrelated**? Justify your answer (**25 pts**).

2. A random variable Y has a mean $\mu_Y = 50$ and a variance $\sigma_Y^2 = 25$. Find an upper bound for the probability that Y deviates from its mean by more than 15, i.e., find an upper bound for $P(|Y - 50| \geq 15)$ (**20 pts**).

3. Imagine a meteorologist states, **"There is a 70% probability of rain tomorrow."** How would a strict **Frequentist** and a strict **Bayesian** interpret this statement differently? What does the "70%" refer to in each philosophy? (**20 pts**)

4. Consider the function $F_X(x) = \frac{x^2}{1+x^2}$. Does this function qualify as a valid CDF? Justify your answer by checking all necessary properties (4 Properties) (**20 pts**).

1. (a)
   - **Independence:** Two random variables are independent if knowing the value of one provides no information about the value of the other, which means their joint PDF is the product of their marginal PDFs.
     Formula: $f_{X,Y}(x, y) = f_X(x)f_Y(y)$.
   - **Uncorrelatedness:** Two random variables are uncorrelated if there is no linear relationship between them, meaning their covariance is zero.
     Formula: $Cov(X, Y) = 0$, which is equivalent to $E[XY] = E[X]E[Y]$.

   (b)
   - **Independence:** No, they are **dependent**. Since $Y = X^2$, the value of Y is completely determined by the value of X. For example, knowing $X = a$ tells us precisely that $Y = a^2$. This demonstrates a complete functional dependence, so they cannot be independent.
   - **Uncorrelatedness:** The variables are uncorrelated if $Cov(X, Y) = 0$.

$$Cov(X, Y) = E[XY] - E[X]E[Y]$$

Since $Y = X^2$, the condition becomes $E[X \cdot X^2] - E[X]E[X^2] = 0$, or:

$$E[X^3] = E[X]E[X^2]$$

We first calculate the required moments for $X \sim U(a, b)$, where the PDF is $f(x) = \frac{1}{b-a}$:
$E[X] = \frac{a+b}{2}$ $\quad$ $E[X^2] = \frac{a^2+ab+b^2}{3}$ $\quad$ $E[X^3] = \frac{(a+b)(a^2+b^2)}{4}$
Now, we substitute these results into the uncorrelatedness condition:

$$\frac{(a+b)(a^2+b^2)}{4} = \left(\frac{a+b}{2}\right)\left(\frac{a^2+ab+b^2}{3}\right)$$

$$\frac{(a+b)(a^2+b^2)}{4} = \frac{(a+b)(a^2+ab+b^2)}{6}$$

This equation is satisfied if either:

i. $(a + b) = 0$, which means $a = -b$. This describes a symmetric interval around zero (e.g., $U(-c, c)$). In this case, the equation becomes $0 = 0$.

ii. Or, if $a + b \neq 0$, we can divide both sides by $(a + b)$:

$$\frac{a^2+b^2}{4} = \frac{a^2+ab+b^2}{6}$$

$$6(a^2 + b^2) = 4(a^2 + ab + b^2)$$

$$6a^2 + 6b^2 = 4a^2 + 4ab + 4b^2$$

$$2a^2 - 4ab + 2b^2 = 0$$

$$a^2 - 2ab + b^2 = 0$$

$$(a - b)^2 = 0 \implies a = b$$

Since a valid uniform distribution requires $a < b$, the second case ($a = b$) is a degenerate distribution (a single point) and is not a valid interval.
Conclusion: X and Y are uncorrelated if and only if the interval is symmetric, $a+b = 0$.

2. Using Chebyshev's inequality, which states that for any $\epsilon > 0$:

$$P(|Y - \mu_Y| \geq \epsilon) \leq \frac{\sigma_Y^2}{\epsilon^2}$$

We are given $\mu_Y = 50$, $\sigma_Y^2 = 25$, and we are evaluating the probability for $\epsilon = 15$:

$$P(|Y - 50| \geq 15) \leq \frac{25}{15^2} = \frac{25}{225} = \frac{1}{9}$$

The upper bound for the probability is $\frac{1}{9}$.

3.
   - A **Frequentist** interprets probability as the long-run relative frequency of an event over many repeated, identical trials. To them, the statement "70% probability of rain" means that if we could observe an infinite number of "tomorrows" with conditions identical to today's, it would rain on 70% of those days. The probability is an objective property of the physical world, and the single event of "tomorrow" will either have rain or not; we just don't know which.
   - A **Bayesian** interprets probability as a subjective degree of belief or confidence in a proposition, given the available evidence. To them, the "70%" represents their personal level of certainty that it will rain tomorrow. This belief is formed by combining prior knowledge (e.g., historical climate data) with current evidence (e.g., satellite imagery, pressure readings) using Bayes' theorem. The probability is a statement about their state of knowledge.

4. No, this is **not a valid CDF** because it violates two of the fundamental properties:

   (a) **Property 1: Limit at $-\infty$ must be 0.** A valid CDF must satisfy $\lim_{x\to-\infty} F_X(x) = 0$. For this function:

   $$\lim_{x\to-\infty} \frac{x^2}{1+x^2} = \lim_{x\to-\infty} \frac{x^2/x^2}{(1/x^2) + (x^2/x^2)} = \lim_{x\to-\infty} \frac{1}{\frac{1}{x^2} + 1} = \frac{1}{0+1} = 1$$

   Since the limit is 1, not 0, it fails this condition.

   (b) **Property 3: Non-decreasing.** A valid CDF must be non-decreasing, i.e., $F_X(x_1) \leq F_X(x_2)$ for any $x_1 < x_2$. We can check this by seeing if its derivative (the PDF, $f_X(x)$) is non-negative ($f_X(x) \geq 0$).

   $$f_X(x) = \frac{d}{dx}\left(\frac{x^2}{1+x^2}\right) = \frac{(2x)(1+x^2) - (x^2)(2x)}{(1+x^2)^2} = \frac{2x + 2x^3 - 2x^3}{(1+x^2)^2} = \frac{2x}{(1+x^2)^2}$$

   For any $x < 0$, the numerator $2x$ is negative, while the denominator $(1+x^2)^2$ is always positive. This means $f_X(x) < 0$ for all $x < 0$, so the function is **decreasing** on the interval $(-\infty, 0)$. This violates the non-decreasing property.

   (Note: It also satisfies Property 2: $\lim_{x\to\infty} F_X(x) = 1$ and Property 4: It is right-continuous. However, since it fails two properties, it is not a valid CDF.)