In the name of GOD.

Sharif University of Technology

Stochastic Process

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Homework 4 | Estimation Theory | Deadline : 1404/10/05

1. [10] Let $X_1, \ldots, X_n$ be independent random variables with pdfs

$$f(x_i \mid \theta) = \begin{cases} \dfrac{1}{2i\theta}, & -i(\theta-1) < x_i < i(\theta+1), \\ 0, & \text{otherwise}, \end{cases}$$

where $\theta > 0$. Find a two-dimensional sufficient statistic for θ.

2. [20] For each of the following distributions let $X_1, \ldots, X_n$ be a random sample. Find a minimal sufficient statistic for θ.

   (a) Location exponential: $f(x \mid \theta) = e^{-(x-\theta)}, \qquad \theta < x < \infty,\ -\infty < \theta < \infty.$

   (b) Cauchy: $f(x \mid \theta) = \frac{1}{\pi\left(1+(x-\theta)^2\right)}, \qquad -\infty < x < \infty,\ -\infty < \theta < \infty.$

3. [20] Let $X_1, \ldots, X_n$ be a random sample from the Uniform$(\theta, \theta+1)$ distribution, where $-\infty < \theta < \infty$.

   (a) Find a minimal sufficient statistic for θ.

   (b) Show that this minimal sufficient statistic is not complete.

4. [30] Let $X_1, \ldots, X_n$ be a random sample from a normal distribution. Denote $S_1 = \sum_{i=1}^n X_i$ and $S_2 = \sum_{i=1}^n X_i^2$. Prove the following statements.

   (a) In the $N(\mu, \mu)$ family, the statistic (S_1, S_2) is sufficient but not minimal sufficient for μ.

   (b) In the $N(\mu, \mu)$ family, the statistic S_2 is minimal sufficient for μ.

   (c) In the $N(\mu, \mu^2)$ family, the statistic (S_1, S_2) is minimal sufficient for μ.

5. [20] Let $X_1, \ldots, X_n$ be a random sample from the inverse Gaussian distribution with pdf

$$f(x \mid \mu, \lambda) = \left(\frac{\lambda}{2\pi x^3}\right)^{1/2} \exp\left\{-\frac{\lambda(x-\mu)^2}{2\mu^2 x}\right\}, \qquad 0 < x < \infty,$$

where $\mu > 0$ and $\lambda > 0$. Show that the statistics

$$\bar{X} = \frac{1}{n}\sum_{i=1}^n X_i \quad \text{and} \quad \frac{1}{\hat{X}} = \frac{n}{\sum_{i=1}^n \frac{1}{X_i} - \frac{1}{\bar{X}}}$$

are sufficient and complete for (μ, λ).

6. [50] Let $X_1, \ldots, X_n$ be i.i.d. with pdf

$$f(x \mid \theta) = \theta x^{\theta-1}, \qquad 0 \le x \le 1,\ 0 < \theta < \infty.$$

   (a) Find the MLE of θ, and show that its variance tends to 0 as $n \to \infty$.

   (b) Find the method of moments estimator of θ.

7. [20] Let $X_1, \ldots, X_n$ be a sample from a population with double exponential (Laplace) pdf

$$f(x \mid \theta) = \frac{1}{2} e^{-|x-\theta|}, \qquad -\infty < x < \infty,\ -\infty < \theta < \infty.$$

   Find the MLE of θ. (Hint: Consider the cases of even n and odd n separately, and express the MLE in terms of the order statistics.)

8. [20] Let X be an observation from the pdf

$$f(x \mid \theta) = \left(\frac{\theta}{2}\right)^{|x|} (1-\theta)^{1-|x|}, \qquad x = -1, 0, 1,\ 0 \le \theta \le 1.$$

   (a) Find the MLE of θ.

   (b) Define the estimator

$$T(X) = \begin{cases} 2, & x = 1, \\ 0, & \text{otherwise.} \end{cases}$$

   Show that $T(X)$ is an unbiased estimator of θ.

   (c) Find a better estimator than $T(X)$ (in the sense of having smaller variance for all θ and strictly smaller for some θ), and prove that it is better.

9. [30] Let $X_1, \ldots, X_N$ be i.i.d. with

$$X_i \sim N(\theta, \sigma^2), \qquad i = 1, \ldots, N,$$

   where σ^2 is known and θ is unknown. Denote $\bar{X} = \frac{1}{N} \sum_{i=1}^{N} X_i$.

   (a) Assume the prior $\theta \sim N(\mu, \sigma^2)$. Show that the posterior density can be written, up to a constant factor, as

$$p(\theta \mid x_1, \ldots, x_N) \propto \exp\left\{ -\frac{1}{2\sigma^2} \left[\sum_{i=1}^{N} (x_i - \theta)^2 + (\theta - \mu)^2 \right] \right\}.$$

   (b) Under the assumptions in part (a), find the MAP estimator of θ.

   (c) Now assume the prior density

$$p(\theta) = \frac{1}{2b} \exp\left(-\frac{|\theta - d|}{b} \right),$$

   and suppose that $\sigma^2 = 2b^2$. Show that, up to an additive constant, the negative log-posterior can be written as

$$L(\theta) = \frac{N}{4b^2} (\bar{X} - \theta)^2 + \frac{1}{b} |\theta - d|.$$

(d) Under the assumptions in part (c), find the MAP estimator of θ by minimizing $L(\theta)$, and show that

$$\hat{\theta}_{\text{MAP}} = \begin{cases} \bar{X} - \frac{2b}{N}, & \bar{X} - d > \frac{2b}{N}, \\ \bar{X} + \frac{2b}{N}, & \bar{X} - d < -\frac{2b}{N}, \\ d, & \text{otherwise.} \end{cases}$$