Solutions – Homework 1 (Review of Probability)

Stochastic Processes – Fall 2025

Problem 1

Suppose A and B are two events with probabilities $P(A) = \frac{2}{3}$ and $P(B) = \frac{1}{2}$.

(a)

Using the formula $P(A \cap B) = P(A) + P(B) - P(A \cup B)$ and the fact that $P(A \cup B) \leq 1$, we get:

$$P(A \cap B) \leq P(A) + P(B) - 1 = \frac{2}{3} + \frac{1}{2} - 1 = \frac{1}{6}$$

Also, $P(A \cap B) \geq 0$ and $P(A \cap B) \leq \min(P(A), P(B)) = \frac{1}{2}$.

Thus, maximum possible value is $\frac{1}{6}$, minimum possible value is 0.

Example for maximum: When $A \cup B$ is the entire sample space and $A \cap B$ is as large as possible.

Example for minimum: When A and B are mutually exclusive.

(b)

Using $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ and $P(A \cap B) \geq 0$, we get:

$$P(A \cup B) \leq P(A) + P(B) = \frac{2}{3} + \frac{1}{2} = \frac{7}{6}$$

But $P(A \cup B) \leq 1$, so maximum is 1.

Minimum occurs when $P(A \cap B)$ is maximum:

$$P(A \cup B) \geq P(A) + P(B) - \min(P(A), P(B)) = \frac{2}{3} + \frac{1}{2} - \frac{1}{2} = \frac{2}{3}$$

Thus, maximum possible value is 1, minimum possible value is $\frac{2}{3}$.

Example for maximum: When one event contains the other.

Example for minimum: When $A \cap B$ is as large as possible.

Problem 2

Suppose n balls are thrown into b bins such that each ball independently falls into one of the bins with equal probability.

(a)

The probability that a specific ball falls into a specific bin is $\frac{1}{b}$.

(b)

Let X_i be an indicator random variable for the i-th ball falling into the given bin. Then:

$$E[X_i] = \frac{1}{b}$$

The expected number of balls in a given bin is:

$$E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i] = \frac{n}{b}$$

(c)

This is a geometric distribution problem. The probability of success (ball falling into given bin) is $p = \frac{1}{b}$.

The expected number of throws until a given bin contains at least one ball is:

$$E[T] = \frac{1}{p} = b$$

(d)

This is the coupon collector problem. Let T_i be the number of additional balls needed to get the i-th new bin after having $i-1$ bins occupied.

Then $T_i \sim \text{Geometric}\left(\frac{b-i+1}{b}\right)$ and:

$$E[T_i] = \frac{b}{b-i+1}$$

The expected number of balls until all bins contain at least one ball is:

$$E[T] = \sum_{i=1}^{b} E[T_i] = \sum_{i=1}^{b} \frac{b}{b-i+1} = b\sum_{j=1}^{b} \frac{1}{j} = bH_b$$

where H_b is the b-th harmonic number.

Problem 3

We prove by induction. For $k = 2$:

$$P(A_1 \cap A_2) = P(A_1)P(A_2|A_1)$$

by definition of conditional probability.

Assume the formula holds for $k-1$:

$$P\left(\bigcap_{i=1}^{k-1} A_i\right) = P(A_1)P(A_2|A_1)\cdots P(A_{k-1}|A_1 \cap \cdots \cap A_{k-2})$$

Then for k:

$$P\left(\bigcap_{i=1}^{k} A_i\right) = P\left(\bigcap_{i=1}^{k-1} A_i\right) P(A_k|A_1 \cap \cdots \cap A_{k-1})$$

Substituting the inductive hypothesis gives the desired result.

Problem 4

(a)

Properties of CDF:

1. $\lim_{x\to-\infty} F(x) = 0$
2. $\lim_{x\to\infty} F(x) = 1$
3. $F(x)$ is non-decreasing
4. $F(x)$ is right-continuous

All properties are satisfied.

[-¿] (-1,0) – (2,0) node[right] x; [-¿] (0,-0.5) – (0,2) node[above] $F(x)$; [thick] (-1,0) – (0,0); [thick] (0,0.5) – (1,1) node[midway,above] $x + \frac{1}{2}$; [thick] (1,1.5) – (2,1.5); [dashed] (0,0) – (0,0.5); [dashed] (1,1) – (1,1.5); (0,0.5) circle (2pt); (1,1) circle (2pt); at (0,0) [below left] 0; at (0.5,0) [below] $\frac{1}{2}$;

(b)

1. $P(0 < X < \frac{1}{4}) = F\left(\frac{1}{4}^-\right) - F(0) = \left(\frac{1}{4} + \frac{1}{2}\right) - \frac{1}{2} = \frac{1}{4}$
2. $P(X = 0) = F(0) - F(0^-) = \frac{1}{2} - 0 = \frac{1}{2}$
3. $P(0 \le X \le \frac{1}{4}) = F\left(\frac{1}{4}\right) - F(0^-) = \left(\frac{1}{4} + \frac{1}{2}\right) - 0 = \frac{3}{4}$

Problem 5

For $p(x)$ to be a valid PMF, we need:

$$\sum_{x=0}^{\infty} p(x) = a \sum_{x=0}^{\infty} \left(\frac{2}{5}\right)^x = a \cdot \frac{1}{1 - \frac{2}{5}} = a \cdot \frac{5}{3} = 1$$

Thus $a = \frac{3}{5}$.

(a)

$$P(X = 2) = \frac{3}{5}\left(\frac{2}{5}\right)^2 = \frac{3}{5} \cdot \frac{4}{25} = \frac{12}{125}$$

(b)

$$P(X \le 2) = \frac{3}{5}\left[1 + \frac{2}{5} + \left(\frac{2}{5}\right)^2\right] = \frac{3}{5}\left[1 + \frac{2}{5} + \frac{4}{25}\right] = \frac{3}{5} \cdot \frac{39}{25} = \frac{117}{125}$$

(c)

$$P(X \ge 1) = 1 - P(X = 0) = 1 - \frac{3}{5} = \frac{2}{5}$$

Problem 6

(a)

For $Y = |X|$, if X has PDF $f_X(x)$, then:

$$f_Y(y) = \begin{cases} f_X(y) + f_X(-y), & y > 0 \\ 0, & \text{otherwise} \end{cases}$$

(b)

For $Y = e^{-X}U(X)$, where $U(X)$ is the unit step function:

$$F_Y(y) = P(Y \le y) = P(e^{-X} \le y) = P(-X \le \ln y) = P(X \ge -\ln y)$$

Thus:

$$f_Y(y) = \frac{d}{dy}F_Y(y) = f_X(-\ln y) \cdot \frac{1}{y}, \quad 0 < y \le 1$$

(c)

For $Y = X^2$:

$$F_Y(y) = P(X^2 \le y) = P(-\sqrt{y} \le X \le \sqrt{y}) = F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

Thus:

$$f_Y(y) = \frac{d}{dy}F_Y(y) = \frac{1}{2\sqrt{y}}[f_X(\sqrt{y}) + f_X(-\sqrt{y})], \quad y > 0$$

Problem 7

(a)

Using the law of total expectation:

$$E[E[X|Y]] = \int E[X|Y=y]f_Y(y)dy = \int\int x f_{X|Y}(x|y)dx f_Y(y)dy = \int x f_X(x)dx = E[X]$$

(b)

Using the definition of variance and law of total expectation:

$$\begin{aligned}
\mathrm{Var}(X) &= E[X^2] - (E[X])^2 \\
&= E[E[X^2|Y]] - (E[E[X|Y]])^2 \\
&= E[\mathrm{Var}(X|Y) + (E[X|Y])^2] - (E[E[X|Y]])^2 \\
&= E[\mathrm{Var}(X|Y)] + E[(E[X|Y])^2] - (E[E[X|Y]])^2 \\
&= E[\mathrm{Var}(X|Y)] + \mathrm{Var}(E[X|Y])
\end{aligned}$$

Problem 8

We want to minimize $E[(X - h(Y))^2]$. For fixed $Y = y$, consider:

$$E[(X - h(y))^2|Y = y] = E[X^2|Y = y] - 2h(y)E[X|Y = y] + h(y)^2$$

Differentiating with respect to $h(y)$:

$$\frac{d}{dh(y)}E[(X - h(y))^2|Y = y] = -2E[X|Y = y] + 2h(y)$$

Setting derivative to zero gives $h(y) = E[X|Y = y]$.

Since this holds for each y, the optimal function is $h(Y) = E[X|Y]$.

Problem 9

Let X and Y be jointly normal with parameters $(\mu_x, \mu_y, \sigma_x^2, \sigma_y^2, \rho)$.

(a)

The marginal distributions are:

$$X \sim N(\mu_x, \sigma_x^2), \quad Y \sim N(\mu_y, \sigma_y^2)$$

(b)

The conditional distribution is:

$$Y|X = x \sim N\left(\mu_y + \rho\frac{\sigma_y}{\sigma_x}(x - \mu_x), \sigma_y^2(1 - \rho^2)\right)$$

(c)

For $Z = aX + bY$, since linear combinations of jointly normal random variables are normal:

$$Z \sim N(a\mu_x + b\mu_y, a^2\sigma_x^2 + b^2\sigma_y^2 + 2ab\rho\sigma_x\sigma_y)$$

Problem 10

Given joint PDF:

$$f_{XY}(x,y) = \begin{cases} \frac{2e^{-2x}}{x}, & 0 \le x < \infty,\ 0 \le y < x \\ 0, & \text{otherwise} \end{cases}$$

First, find marginal PDF of X:

$$f_X(x) = \int_0^x \frac{2e^{-2x}}{x} dy = 2e^{-2x}, \quad x > 0$$

So $X \sim \text{Exp}(2)$ and $E[X] = \frac{1}{2}$, $E[X^2] = \frac{1}{2}$.
Now find $E[Y]$:

$$E[Y] = \int_0^\infty \int_0^x y \cdot \frac{2e^{-2x}}{x} dydx = \int_0^\infty \frac{2e^{-2x}}{x} \cdot \frac{x^2}{2} dx = \int_0^\infty xe^{-2x} dx = \frac{1}{4}$$

Find $E[XY]$:

$$E[XY] = \int_0^\infty \int_0^x xy \cdot \frac{2e^{-2x}}{x} dydx = \int_0^\infty \frac{2e^{-2x}}{x} \cdot \frac{x^3}{2} dx = \int_0^\infty x^2 e^{-2x} dx = \frac{1}{4}$$

Thus:

$$\text{Cov}(X,Y) = E[XY] - E[X]E[Y] = \frac{1}{4} - \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$$

Problem 11

Let X_i be indicator random variable for person i getting their own hat back.

(a)

$$P(X_i = 1) = \frac{1}{n}, \quad E[X_i] = \frac{1}{n}$$

By linearity of expectation:

$$E[X] = \sum_{i=1}^n E[X_i] = n \cdot \frac{1}{n} = 1$$

(b)

Variance: $\text{Var}(X) = \sum_{i=1}^n \text{Var}(X_i) + 2\sum_{i<j} \text{Cov}(X_i, X_j)$
$\text{Var}(X_i) = \frac{1}{n}\left(1 - \frac{1}{n}\right)$
For $i \neq j$:

$$P(X_i = 1, X_j = 1) = \frac{1}{n(n-1)}$$

$$\text{Cov}(X_i, X_j) = E[X_iX_j] - E[X_i]E[X_j] = \frac{1}{n(n-1)} - \frac{1}{n^2} = \frac{1}{n^2(n-1)}$$

Thus:

$$\text{Var}(X) = n \cdot \frac{1}{n}\left(1 - \frac{1}{n}\right) + 2\binom{n}{2} \cdot \frac{1}{n^2(n-1)} = 1 - \frac{1}{n} + 1 = 1$$

(c)

For large n, the distribution of X approaches Poisson(1). This is the limiting distribution of the number of fixed points in a random permutation.