In the name of GOD.
Sharif University of Technology
Stochastic Processes Fall 2025 Hamid R. Rabiee

Homework 1 Review of Probability Due: Nov 2

1. Suppose A and B are two events with probabilities $P(A) = \frac{2}{3}$ and $P(B) = \frac{1}{2}$.
(a) What is the maximum possible value of $P(A \cap B)$? What is the minimum possible value? Give an example for each case.
(b) What is the maximum possible value of $P(A \cup B)$? What is the minimum possible value? Give an example for each case.

2. Suppose n balls are thrown into b bins such that each ball independently falls into one of the bins with equal probability.
(a) What is the probability that a specific ball falls into a specific bin?
(b) What is the expected number of balls in a given bin?
(c) What is the expected number of balls that must be thrown until a given bin contains at least one ball?
(d) What is the expected number of balls that must be thrown until all bins contain at least one ball?
Assume $n \gg b$ for parts (c) and (d).

3. If $A_1, A_2, \ldots, A_k$ are events such that $P(A_1 \cap A_2 \cap \cdots \cap A_k) > 0$, show that

$$P\left(\bigcap_{i=1}^{k} A_i\right) = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2) \cdots P(A_k|A_1 \cap A_2 \cap \cdots \cap A_{k-1}).$$

4. Consider the function

$$F(x) = \begin{cases} 0, & x < 0, \\ x + \frac{1}{2}, & 0 \leq x < \frac{1}{2}, \\ 1, & x \geq \frac{1}{2}. \end{cases}$$

(a) Plot $F(x)$ and show that it satisfies the properties of a CDF.
(b) If X is a random variable with this CDF $F(x)$, find:
   1. $P(0 < X < \frac{1}{4})$
   2. $P(X = 0)$
   3. $P(0 \leq X \leq \frac{1}{4})$

5. Show that the function

$$p(x) = \begin{cases} a\left(\frac{2}{5}\right)^x, & x = 0, 1, 2, \ldots, \\ 0, & \text{otherwise}, \end{cases}$$

is a valid PMF for random variable X (by finding a). Then compute:
(a) $P(X = 2)$
(b) $P(X \leq 2)$
(c) $P(X \geq 1)$

6. Find the distribution $f_Y(y)$ of $Y = g(X)$ in terms of $f_X(x)$ for:
(a) $g(x) = |x|$
(b) $g(x) = e^{-x}U(x)$
(c) $g(x) = x^2$

7. For any two random variables X and Y, show that:
(a) $E[E[X|Y]] = E[X]$
(b) $\text{Var}(X) = E[\text{Var}(X|Y)] + \text{Var}(E[X|Y])$

8. Let X and Y be random variables. Show that the function h minimizing $E[(X - h(Y))^2]$ is:
$$h(y) = E[X|Y = y],$$
assuming $E[X^2] < \infty$.

9. Let X and Y be jointly normal random variables with parameters $(\mu_x, \mu_y, \sigma_x^2, \sigma_y^2, \rho)$. Derive:
(a) The marginal distributions of X and Y.
(b) The conditional distribution of Y given $X = x$.
(c) The distribution of $aX + bY$ for constants a, b.

10. If X and Y have joint PDF
$$f_{XY}(x, y) = \begin{cases} \frac{2e^{-2x}}{x}, & 0 \leq x < \infty,\ 0 \leq y < x, \\ 0, & \text{otherwise}, \end{cases}$$
compute $\text{Cov}(X, Y)$.

11. At a seminar, n people each throw their hat into a box. The hats are then mixed up, and each person randomly draws one hat from the box. Let X be the number of people who get their own hat back.

    (a) Find $E[X]$. (Use the linearity of expectation and define appropriate indicator random variables.)

    (b) Find $\text{Var}(X)$. (*Hint:* Consider the covariance between your indicator variables.)

    (c) What is the distribution of X for large n? (You don't need to calculate the exact PMF, but describe its limiting behavior.)