Stochastic Processes

Week 09 (Version 1.0)

Markov Chains & HMMs

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Overview

Markov Property

Markov Chains Definition

Markov Chains Stationary Property

Markov Chains Paths

Markov Chains Classification of States

Markov Chains Steady States

Hidden Markov Models

Markov Property

- A discrete process has the Markov property if given its value at time t, the value at time t+1 is independent of values at times before t.

That is:

$$Pr(X_{t+1} = x_{t+1} | X_t = x_t, X_{t-1} = x_{t-1}, \dots, X_1 = x_1)$$
$$= Pr(X_{t+1} = x_{t+1} | X_t = x_t)$$

For all t, $\mathrm{x}_{\mathrm{t}+1}, \mathrm{x}_{\mathrm{t}}, x_{t-1}, x_{t-2}, \dots, x_1$.

Stationary Property

- A Markov Process is called stationary if:

$$\Pr\left(X_{t+1} = u | X_t = v\right) = \Pr\left(X_1 = u | X_0 = v\right) \text{ for all } t.$$

- The evolution of stationary processes don't change over time.
- For defining the complete joint distribution of a stationary Markov Process it is sufficient to define $\Pr(X_1 = u | X_0 = v)$ and $\Pr(X_0 = v)$ for all u and v.
- We will mainly consider stationary Markov processes here.

Markov Process Types

- There exist two types of Markov processes based on domain of X_t values:
  - Discrete
  - Continuous

- Discrete Markov processes are called "Markov Chains" (MC).

Markov Process Types

State Space	Type of Parameter: Discrete	Type of Parameter: Continuous
Discrete	Discrete-Time Markov Chain	Continuous-Time Markov Chain
Continuous	Discrete-Time Markov Process	Continuous-Time Markov Process

- In this course we will focus on stationary MCs.

Example (Coin Tossing Game)

- Consider a single player game in which at every step a biased coin is tossed and according to the result, the score will be increased or decreased by one point.
- The game ends if either the score reaches 100 (winning) or -100 (losing).
- Score of the player at each step $t \geq 0$ is a random variable and the sequence of scores as the game progresses forms a random process $X_0, X_1, \ldots, X_t$.

Example (Coin Tossing Game)

- It is easy to verify that X is a stationary Markov chain: At the end of each step the score solely depends on the current score s_c and the result of tossing the coin (which is independent of time and previous tosses).
- Stating this mathematically (for $s_c \notin \{-100,100\}$):

$$Pr(X_{t+1} = s|X_t = s_c, X_{t-1} = s_{t-1}, \dots, X_0 = 0)$$

$$= \begin{cases} p & ; s = s_c + 1 \\ 1 - p & ; s = s_c - 1 \\ 0 & ; otherwise \end{cases}$$

Independent of t and $s_0, \dots, s_{t-1}$

$$= Pr(X_{t+1} = s|X_t = s_c) = Pr(X_1 = s|X_0 = s_c)$$

- If value of p was subject to change in time, the process would not be stationary (in the formulation we would have p_t instead of p).

Transition matrix

- According to the Markov property and stationary property, at each time step the process moves according to a fixed transition matrix:

$$P(X_{t+1} = j | X_t = i) = p_{ij}$$

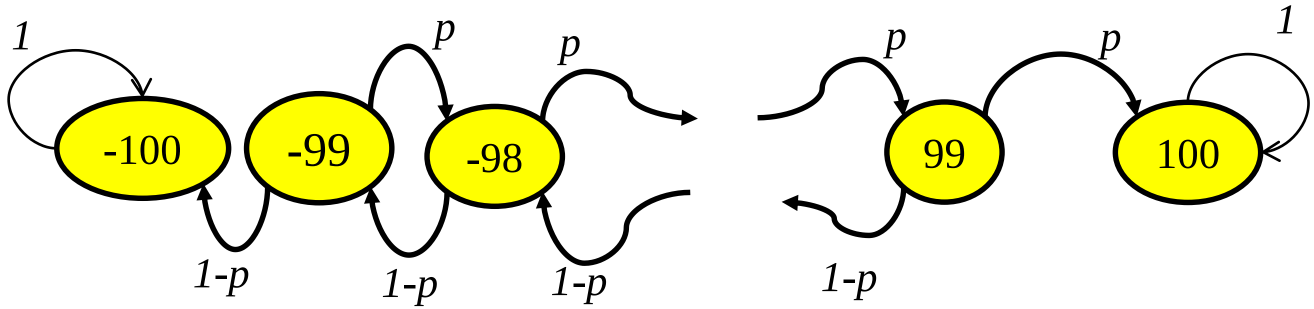
- Stochastic matrix: Rows sum up to 1.
  Double stochastic matrix: Rows and columns sum up to 1.

State Graph

- It is convenient to visualize a stationary Markov Chain with a transition diagram:
  - A node represents a possible value of X_t (state). At each time t, we say the process is in state s if X_t=s.
  - Each edge represents the probability of going from one state to another (we omit edges with zero weight).
  - We should also specify the vector of initial probabilities $\pi = (\pi_1, \ldots, \pi_n)$ where $\pi_i = \Pr(X_0 = i)$.
- A stationary discrete process could be described as a person walking randomly on a graph (considering each step to depend only on the state he/she is currently in). The resulted path is called a "Random Walk".

Example

- The transition diagram of the coin tossing game is:

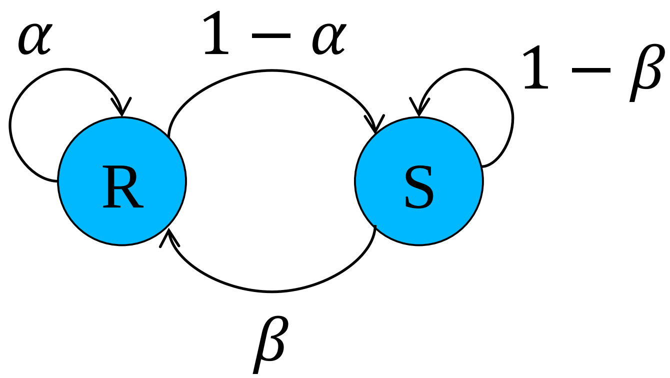


- We modeled winning and losing by states which when we get into, we never get out.
- Note that if the process was not stationary we were not able to visualize it in this way: For example consider the case that p is changing in time.

Example 1 (Modeling Weather)

- Example: Assume each day is sunny or rainy. If a day is rainy, the next day is rainy with probability α (and sunny with probability $1-\alpha$). If the day is sunny, the next day is rainy with probability β (and sunny with probability $1-\beta$).

S = {rainy, sunny}, $P = \begin{bmatrix} \alpha & 1-\alpha \\ \beta & 1-\beta \end{bmatrix}$



Examplem 2 (Modeling Weather)

Suppose that whether or not it rains today depends on previous weather conditions through the last two days.

Suppose that:

- if it has rained for the past two days, then it will rain tomorrow with probability 0.7
- if it rained today but not yesterday, then it will rain tomorrow with probability 0.5
- if it rained yesterday but not today, then it will rain tomorrow with probability 0.4
- if it has not rained in the past two days, then it will rain tomorrow with probability 0.2.

Examplem 2 (Modeling Weather)

If we let the state at time n depend only on whether or not it is raining at time n, then the preceding model is not a Markov chain.

We can transform this model into a Markov chain by saying that the state at any time is determined by the weather conditions during both that day and the previous day:

state 0 if it rained both today and yesterday,
state 1 if it rained today but not yesterday,
state 2 if it rained yesterday but not today,
state 3 if it did not rain either yesterday or today.

Example 2 (Modeling Weather)

	t-1	t	t+1 (p(R))
S_0	R	R	0.7
S_1	S	R	0.5
S_2	R	S	0.4
S_3	S	S	0.2

$$
\begin{array}{c c}
 & \begin{array}{cccc} S_0 & S_1 & S_2 & S_3 \end{array} \\
\begin{array}{c} S_0 \\ S_1 \\ S_2 \\ S_3 \end{array} & \begin{bmatrix} 0.7 & 0 & 0.3 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.4 & 0 & 0.6 \\ 0 & 0.2 & 0 & 0.8 \end{bmatrix}
\end{array}
$$

The Chapman-Kolmogorov Equation

- Define the n-step transition $p_{ij}^{(n)}$ as the probability that starting from state *i*, the process stops at state *j* after *n* time steps:

$$p_{ij}^{(n)} = \mathrm{P}\{\mathrm{X}_{\mathrm{n+m}} = j \mid \mathrm{X}_{\mathrm{m}} = i\}$$

- Then the Chapman-Kolomogorov equation is given by:

$$p_{ij}^{(n+m)} = \sum_{k=0}^{\infty} p_{ik}^{(n)} p_{kj}^{(m)}$$

- Corollary 1: $P^{(n)}$ can be calculated by: $P^{(n)} = P^n$
- Corollary 2: If the process starts at time 0 with distribution π on the states then after n steps the distribution is πP^n.

The Chapman-Kolmogorov Equation

$$P_{ij}^{n+m} = P(X_{n+m} = j | X_0 = i)$$

$$= \sum_{k=1}^{v} P(X_{n+m} = j, X_n = k | X_0 = i)$$

$$= \sum_{k=1}^{v} P(X_{n+m} = j | X_n = k) P(X_n = k | X_0 = i)$$

$$= \sum_{k=1}^{v} P_{ik}^{n} P_{kj}^{m} = (P^m P^n)(ij)$$

The Chapman-Kolmogorov Equation

- Corollary 1: $P^{(n)}$ can be calculated by: $P^{(n)} = P^n$

$$P^{(2)} = P^{(1+1)} = P^{(1)}P^{(1)} = P \cdot P = P^2$$

$$P^{(n)} = P^{(n-1+1)} = P^{(n-1)}P^{(1)} = P^{n-1}P = P^n$$

$$P^{(1)} = \begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix}$$

$$P^{(2)} = P^2 = \begin{pmatrix} 0.61 & 0.39 \\ 0.52 & 0.48 \end{pmatrix}$$

$$P^{(4)} = P^2P^2 = \begin{pmatrix} 0.57 & 0.43 \\ 0.57 & 0.43 \end{pmatrix}$$

Example 2 (Modeling Weather)

- If **Monday** and **Tuesday** are raining, what is the probability of raining on **Thursday**?

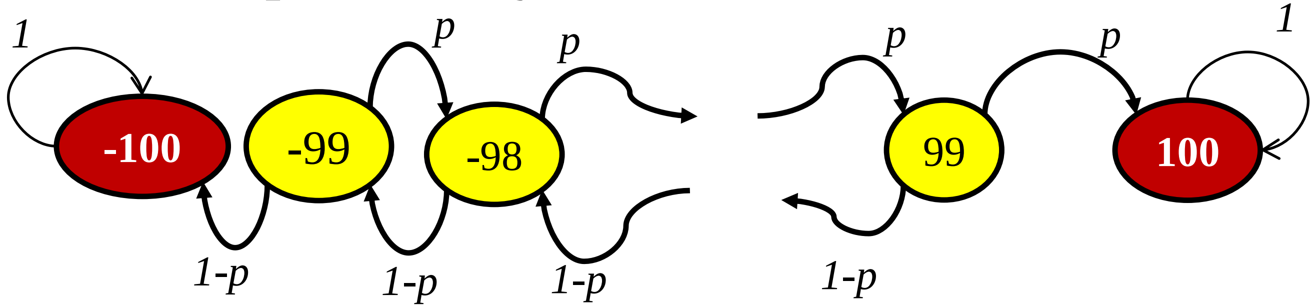
$$\begin{array}{c} \\ S_0 \\ S_1 \\ S_2 \\ S_3 \end{array} \begin{array}{c} \begin{matrix} S_0 & S_1 & S_2 & S_3 \end{matrix} \\ \begin{bmatrix} 0.7 & 0 & 0.3 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.4 & 0 & 0.6 \\ 0 & 0.2 & 0 & 0.8 \end{bmatrix} \end{array}$$

$$P^2 = P^2 = \begin{bmatrix} 0.7 & 0 & 0.3 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.4 & 0 & 0.6 \\ 0 & 0.2 & 0 & 0.8 \end{bmatrix}^2 = \begin{bmatrix} 0.49 & 0.12 & 0.21 & 0.18 \\ 0.35 & 0.2 & 0.15 & 0.3 \\ 0.2 & 0.12 & 0.2 & 0.48 \\ 0.1 & 0.16 & 0.1 & 0.64 \end{bmatrix}$$

Since rain on Thursday is equivalent to the process being in either state 0 or state 1 on Thursday, the desired probability is given by $P^2_{00} + P^2_{01} = 0.49 + 0.12 = 0.61$.

Absorbing Markov Chain

- An absorbing state is one in which the probability that the process remains in that state once it enters the state is 1 (i.e., $p_{ii} = 1$).
- A Markov chain is absorbing if it has at least one absorbing state, and if from every state it is possible to go to an absorbing state (not necessarily in one step).
- An absorbing Markov chain will eventually enter one of the absorbing states and never leave it.
- Example: The 100 and -100 states in coin tossing game (Note: After playing long enough, the player will either win or lose with probability 1.



Absorption Theorem

- In an absorbing Markov chain the probability that the process will be absorbed is 1.
- Proof: From each non-absorbing state s_j it is possible to reach an absorbing state starting from s_j. Therefore there exists *p* and *m*, such that the probability of not absorbing after *m* steps is at most *p*, in 2*m* steps at most p^2, etc.
- Since the probability of not being absorbed is monotonically decreasing, we have:

$$\lim_{n\to\infty} \text{P}(\text{not absorbed after } n \text{ steps}) = 0$$

Classification of States

- Accessibility: State j is said to be accessible from state i if starting in i it is possible that the process will ever enter state j: $(P^n)_{ij} > 0$.
- Communication: Two states i and j that are accessible to each other are said to communicate.
  - Every node communicates with itself:
    $$p_{ii}^{(0)} = \mathrm{P}\,(X_0 = i | X_0 = i) = 1$$
  - Communication is an equivalence relation: It divides the state space up into a number of separate classes in which every pair of states communicate.
- The Markov chain is said to be irreducible if it has only one class.

Transient and Recurrent states

- For any state *i* we let f_i denote the probability that, starting in state *i*, the process will ever reenter state *i*:

$$f_i = \Pr(\exists n : X_n = i \mid X_0 = i)$$

- State *i* is said to be recurrent if $\mathrm{f}_i = 1$ and transient if $\mathrm{f}_i < 1$.
- **Theorem 1:** State *i* is recurrent if and only if, starting in state *i*, the expected number of time periods that the process is in state *i* is infinite:
- **Corollary 1:** A transient state will only be visited a finite number of times.

Proof: $E[size(\{n: X_n = i\})|X_0 = i]$

$$= \sum_{k=1}^{\infty} k \times Pr(size(\{n: X_n = i\}) = k|X_0 = i)$$

$$= \ldots + \infty \times prob(size(\{n: X_n = i\}) = \infty|X_0 = i) < \infty$$

$$\Rightarrow prob(size(\{n: X_n = i\}) = \infty|X_0 = i) = 0$$

Transient and Recurrent states

- **Theorem 2:** State i is recurrent iff $\sum_{n=1}^{\infty} (P^n)_{ii} = \infty$.

  (Look at the reference book for proof).

- **Corollary 2:** A finite state Markov chain has at least one recurrent state.

  If all states are transient there will be a finite number of steps that after that the process should not be in any state (which is a contradiction).

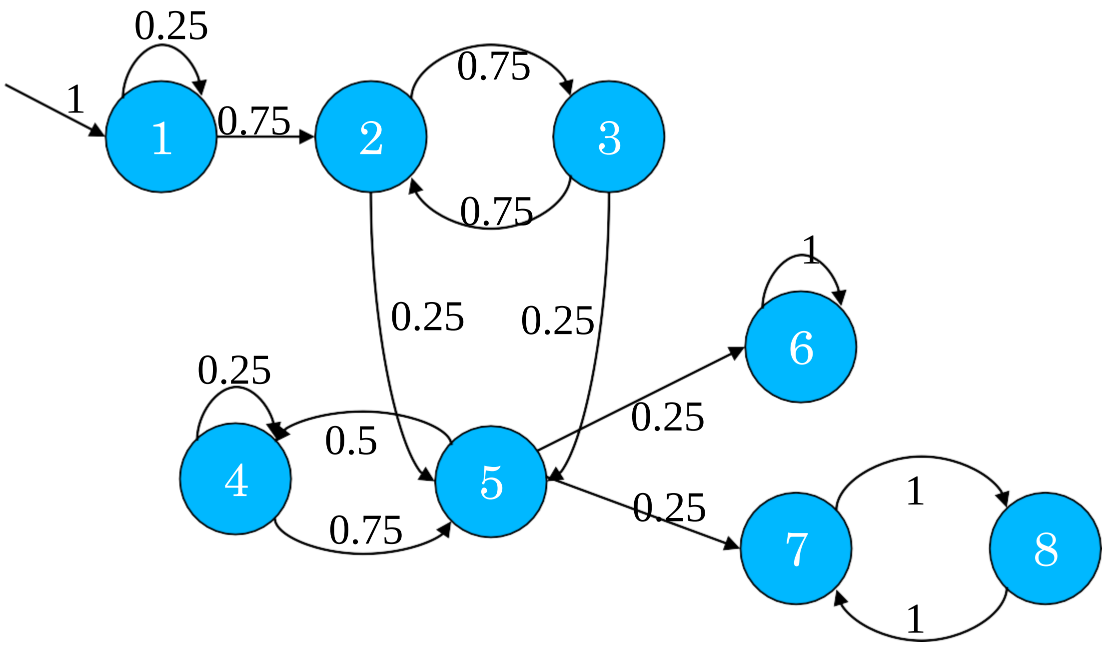
Ergodic States

- If state i is recurrent, then it is said to be positive recurrent if, starting in i, the expected time until the process returns to state i is finite.
- In a finite-state MC, all recurrent states are positive recurrent.
- State i is said to have period $d(i)$ if $(p^n)_{ii}=0$ whenever n is not divisible by d, and d is the largest integer with this property.
- Equivalently: $d = \gcd\{n: \Pr(X_n = i \mid X_0 = i) > 0\}$
- A state with period 1 is said to be aperiodic.
- We call an MC aperiodic if all its states are aperiodic.

Ergodic States

- A state *i* is said to be ergodic if it is aperiodic and positive recurrent.
- Period, recurrence and positive recurrence are all class properties. They are shared between states of a class.

Example



$Classes$: {1}, {2,3}, {4,5}, {6}, {7,8}

$Recurrent\ states$: 6,7,8

$Absorbing\ states$: 6

$Ergodic\ states$: 6

$Periodic\ states$: 2, 3, 7, 8: $\quad Period$ 2

Example

As time goes to infinity, what is the probability of being in each class?

Answer:

- The process will be in transient classes {1},{2,3},{4,5} with probability 0.
- Problem is symmetric for entering classes {6} and {7,8} as their only input edge is one from 5 with equal probabilities 0.25, and once it enters them, there is no way out.
- Therefore, at infinity probability of being in each of these two classes is 0.5.

Example

If the process is absorbed in {7,8} (which could be considered as an absorbing super state) what will happen after that?

Answer:

- It will alternate between 7 and 8 to the end. Therefore, at time $t \to \infty$ probability of being in 7 (or 8) will depend on the parity of t. In general finding the exact behavior of non-ergodic states as $t \to \infty$ is not easy.

Steady State

Theorem: For an irreducible ergodic Markov chain $\lim_{n\to\infty}(P^n)_{ij}$ exists and is independent of i. Furthermore, letting:

$$\pi_j^* = \lim_{n\to\infty}(P^n)_{ij}$$

Then $\pi^* = (\pi_1^*, \dots \pi_d^*)^t$ is unique nonnegative solution of:

$$\begin{cases} \pi^* = \pi^* P \\ \sum_{j=1}^{d} \pi_j = 1 \end{cases}$$

- If the ergodicity condition is removed, $\lim_{n\to\infty}(P^n)_{ij}$ does not exist in general, but the given equations yet have a unique solution $\pi^* = (\pi_1^*, \dots \pi_d^*)^t$ in which π_j^* will be equal to the long run proportion of time that the Markov chain is in state j.

Example

- Consider the weather model example discussed before. We want to see how will the weather be when time goes to infinity:

$$P = \begin{bmatrix} \alpha & 1-\alpha \\ \beta & 1-\beta \end{bmatrix}$$

$$\begin{cases} \pi_0^* = \alpha\pi_0^* + \beta\pi_1^* \\ \pi_1^* = (1-\alpha)\pi_0^* + (1-\beta)\pi_1^* \\ \pi_0^* + \pi_1^* = 1 \end{cases}$$

One of these equations is redundant. (why?)

- Which yields that $\pi_0^* = \frac{\beta}{1+\beta-\alpha}$ and $\pi_1^* = \frac{1-\alpha}{1+\beta-\alpha}$.
- Exercise: In each of the following cases investigate the existence of solution and its meaning:
  - 1) α=0 and $\beta = 1$
  - 2) α=1 and $\beta = 0$

Introduction to Hidden Markov Models

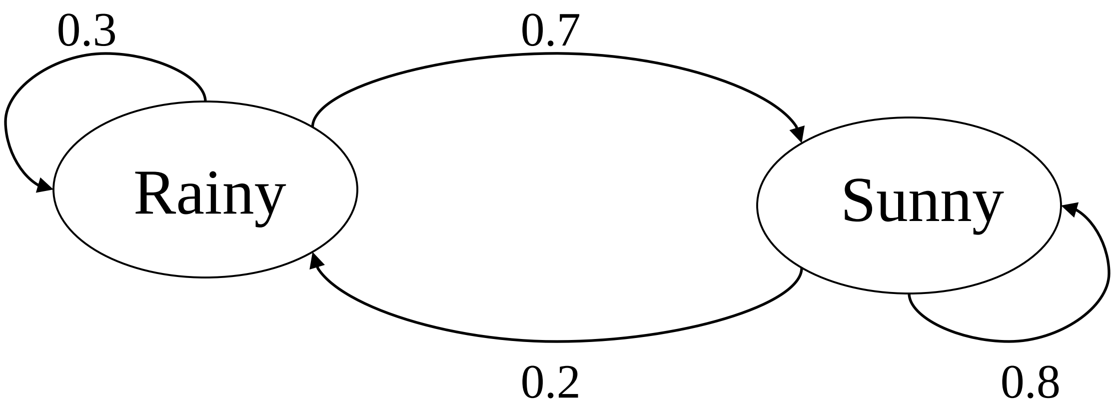
Markov Models

- Set of states: $\{s_1, s_2, \ldots, s_N\}$
- Process moves from one state to another generating a sequence of states : $s_{i1}, s_{i2}, \ldots, s_{ik}, \ldots$
- Markov chain property: probability of each subsequent state depends only on what was the previous state:

$$P(s_{ik} \mid s_{i1}, s_{i2}, \ldots, s_{ik-1}) = P(s_{ik} \mid s_{ik-1})$$

- To define a Markov model, the following probabilities have to be specified: transition probabilities $a_{ij} = P(s_i \mid s_j)$ and initial probabilities $\pi_i = P(s_i)$

Example of Markov Model



- Two states : ‘Rainy’ and ‘Sunny’.
- Transition probabilities: P(‘Rainy’|‘Rainy’)=0.3 , P(‘Sunny’|‘Rainy’)=0.7 , P(‘Rainy’|‘Sunny’)=0.2, P(‘Sunny’|‘Sunny’)=0.8
- Initial probabilities: say P(‘Rainy’)=0.4 , P(‘Sunny’)=0.6 .

Hidden Markov models.

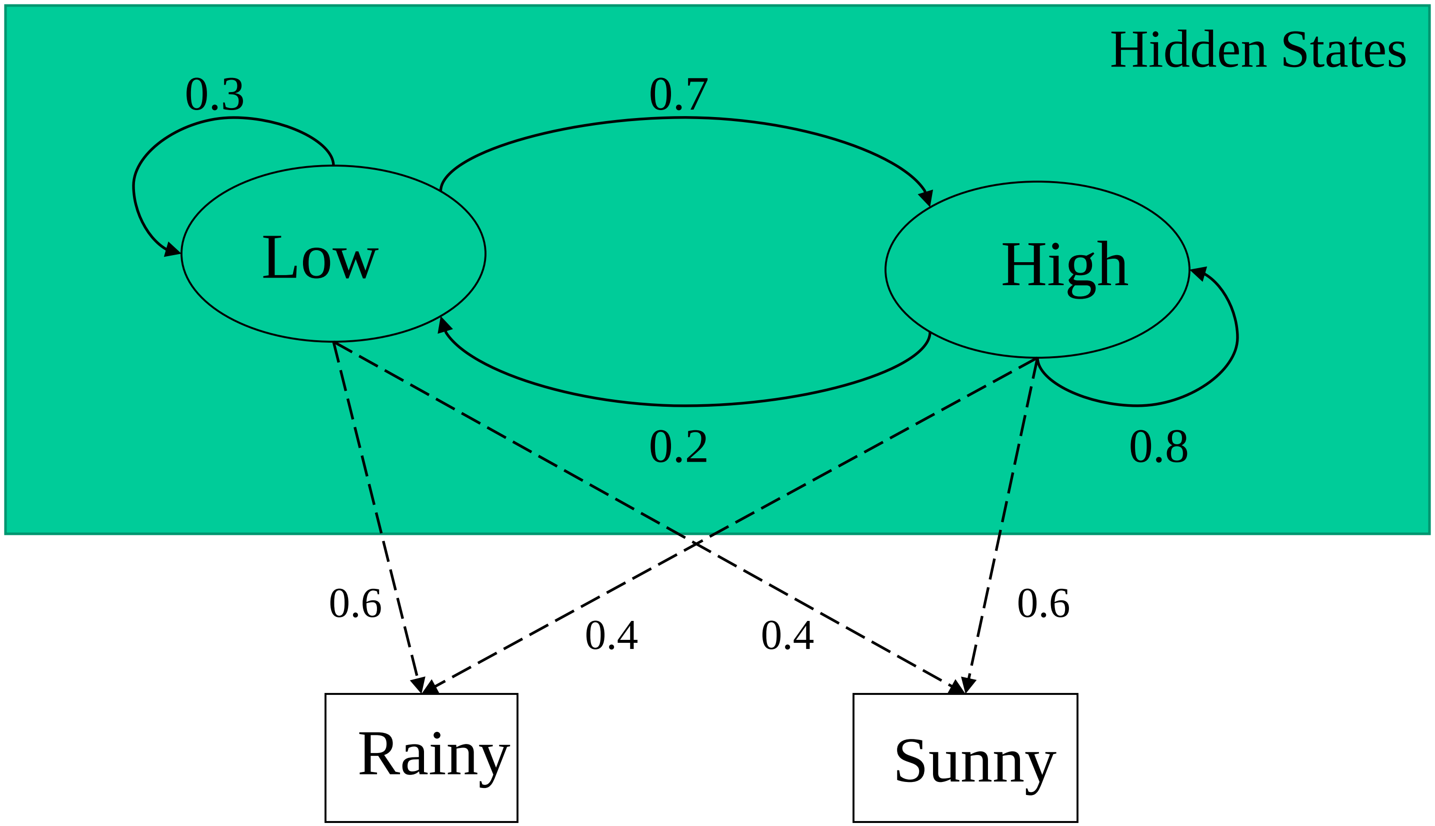
- Set of states: $\{s_1, s_2, \ldots, s_N\}$
- Process moves from one state to another generating a sequence of states : $s_{i1}, s_{i2}, \ldots, s_{ik}, \ldots$
- Markov chain property: probability of each subsequent state depends only on what was the previous state:

$$P(s_{ik} \mid s_{i1}, s_{i2}, \ldots, s_{ik-1}) = P(s_{ik} \mid s_{ik-1})$$

- States are not visible, but each state randomly generates one of M observations (or visible states) $\{v_1, v_2, \ldots, v_M\}$

- To define hidden Markov model, the following probabilities have to be specified: matrix of transition probabilities $A=(a_{ij})$, $a_{ij}= P(s_i \mid s_j)$, matrix of observation probabilities $B=(b_i(v_m))$, $b_i(v_m)= P(v_m \mid s_i)$ and a vector of initial probabilities $\pi=(\pi_i)$, $\pi_i = P(s_i)$. Model is represented by $M=(A, B, \pi)$.

Example of Hidden Markov Model



Example of Hidden Markov Model

- Two states : ‘Low’ and ‘High’ atmospheric pressure.
- Two observations : ‘Rainy’ and ‘Sunny’.
- Transition probabilities: $P(\text{‘Low’}|\text{‘Low’})=0.3$, $P(\text{‘High’}|\text{‘Low’})=0.7$, $P(\text{‘Low’}|\text{‘High’})=0.2$, $P(\text{‘High’}|\text{‘High’})=0.8$
- Observation probabilities : $P(\text{‘Rainy’}|\text{‘Low’})=0.6$, $P(\text{‘Sunny’}|\text{‘Low’})=0.4$, $P(\text{‘Rainy’}|\text{‘High’})=0.4$, $P(\text{‘Sunny’}|\text{‘High’})=0.3$.
- Initial probabilities: say $P(\text{‘Low’})=0.4$, $P(\text{‘High’})=0.6$.

Calculation of observation sequence probability

- Suppose we want to calculate a probability of a sequence of observations in our example, {‘Sunny’,’Rainy’}.
- Consider all possible hidden state sequences:

P({‘Sunny’,’Rainy’}) = P({‘Sunny’,’Rainy’} , {‘Low’,’Low’}) + P({‘Sunny’,’Rainy’} , {‘Low’,’High’}) + P({‘Sunny’,’Rainy’} , {‘High’,’Low’}) + P({‘Sunny’,’Rainy’} , {‘High’,’High’})

where first term is :

P({‘Sunny’,’Rainy’} , {‘Low’,’Low’})=
P({‘Sunny’,’Rainy’} | {‘Low’,’Low’}) P({‘Low’,’Low’}) =
P(‘Sunny’|’Low’)P(‘Rainy’|’Low’) P(‘Low’)P(‘Low’|’Low)
= 0.4*0.4*0.6*0.4*0.3

Main issues using HMMs :

Evaluation problem. Given the HMM $M=(A, B, \pi)$ and the observation sequence $O=o_1 o_2 \ldots o_K$, calculate the probability that model M has generated sequence O.

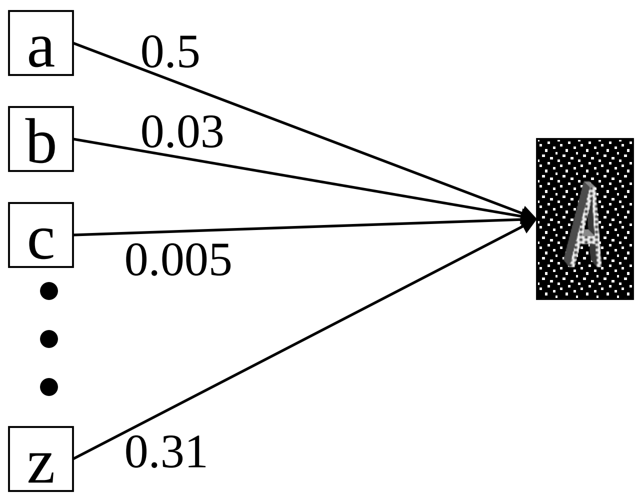
- **Decoding problem.** Given the HMM $M=(A, B, \pi)$ and the observation sequence $O=o_1 o_2 \ldots o_K$, calculate the most likely sequence of hidden states S_i that produced this observation sequence O.
- **Learning problem.** Given some training observation sequences $O=o_1 o_2 \ldots o_K$ and general structure of HMM (numbers of hidden and visible states), determine HMM parameters $M=(A, B, \pi)$ that best fit training data.

$O=o_1 \ldots o_K$ *denotes a sequence of observations* $o_k \in \{v_1, \ldots, v_M\}$.

Word recognition example(1).

- Typed word recognition, assume all characters are separated.

- Character recognizer outputs probability of the image being particular character, P(image|character).



Hidden state Observation

Word recognition example(2).

• Hidden states of HMM = characters.

• Observations = typed images of characters segmented from the image v_α . Note that there is an infinite number of observations

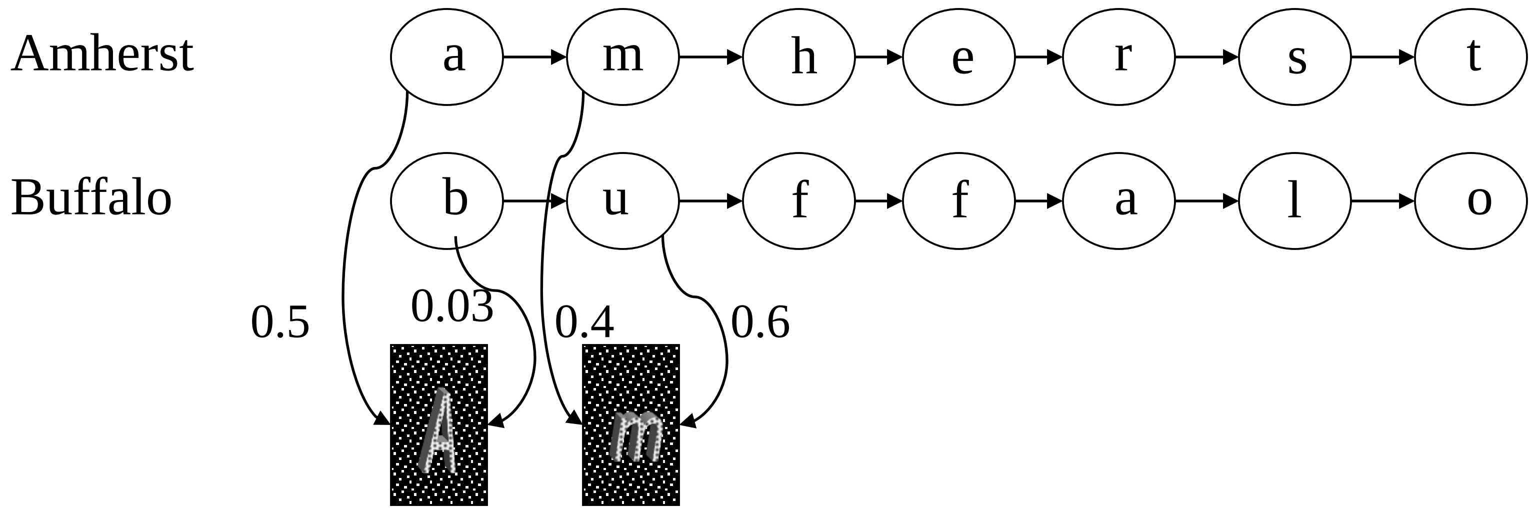
• Observation probabilities = character recognizer scores.

$$B = \left(b_i(v_\alpha)\right) = \left(P(v_\alpha \mid s_i)\right)$$

•Transition probabilities will be defined differently in two subsequent models.

Word recognition example(3).

• If lexicon is given, we can construct separate HMM models for each lexicon word.

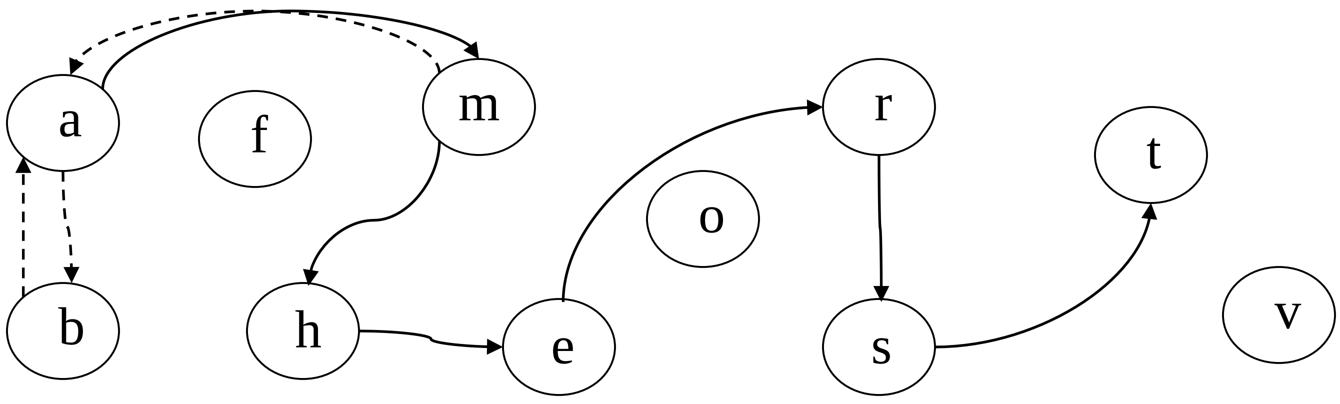


• Here recognition of word image is equivalent to the problem of evaluating few HMM models.

•This is an application of **Evaluation problem.**

Word recognition example(4).

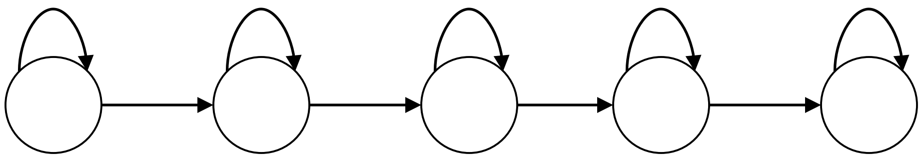
- We can construct a single HMM for all words.
- Hidden states = all characters in the alphabet.
- Transition probabilities and initial probabilities are calculated from language model.
- Observations and observation probabilities are as before.



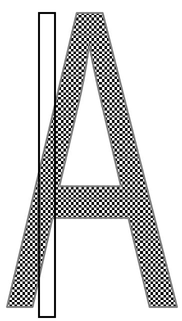
- Here we have to determine the best sequence of hidden states, the one that most likely produced word image.
- This is an application of **Decoding problem.**

Character recognition with HMM example.

- The structure of hidden states is chosen.



- Observations are feature vectors extracted from vertical slices.



- Probabilistic mapping from hidden state to feature vectors:
    1. use mixture of Gaussian models
    2. Quantize feature vector space.

Evaluation Problem.

•**Evaluation problem.** Given the HMM $M=(A, B, \pi)$ and the observation sequence $O=o_1 o_2 \ldots o_K$, calculate the probability that model M has generated sequence O.

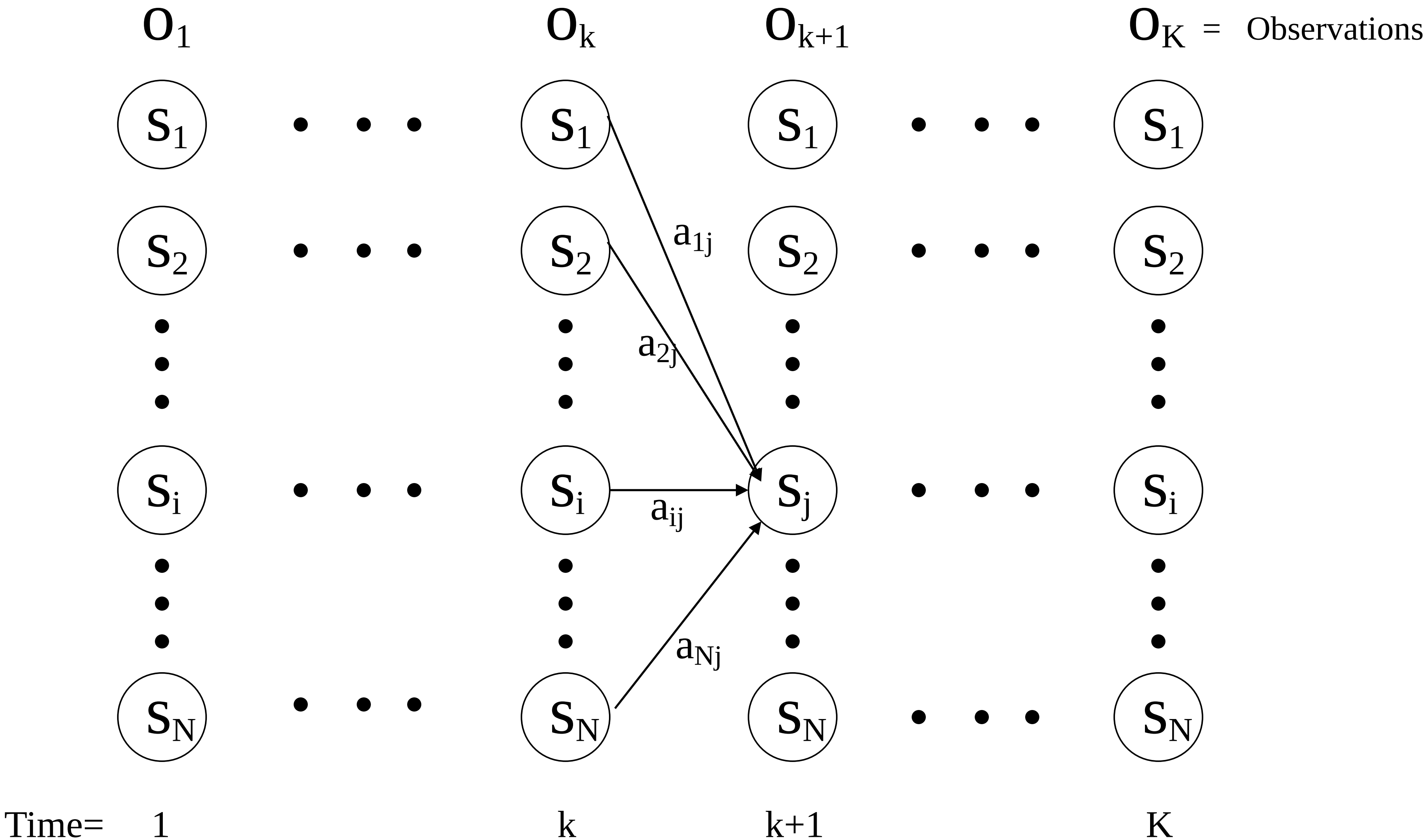
• Trying to find probability of observations $O=o_1 o_2 \ldots o_K$ by means of considering all hidden state sequences (as was done in example) is impractical:

N^K hidden state sequences - exponential complexity.

• Use **Forward-Backward HMM algorithms** for efficient calculations.

• Define the forward variable $\alpha_k(i)$ as the joint probability of the partial observation sequence $o_1 o_2 \ldots o_k$ and that the hidden state at time k is s_i : $\alpha_k(i)= P(o_1 o_2 \ldots o_k , q_k= s_i)$

Trellis representation of an HMM



Forward recursion for HMM

- Initialization:

$$\alpha_1(i) = P(o_1, q_1 = s_i) = \pi_i\, b_i(o_1)\,, \quad 1 <= i <= N.$$

- Forward recursion:

$$\alpha_{k+1}(i) = P(o_1\, o_2 \ldots\, o_{k+1}, q_{k+1} = s_j) =$$
$$\Sigma_i\, P(o_1\, o_2 \ldots\, o_{k+1}, q_k = s_i, q_{k+1} = s_j) =$$
$$\Sigma_i\, P(o_1\, o_2 \ldots\, o_k, q_k = s_i)\, a_{ij}\, b_j(o_{k+1}) =$$
$$[\Sigma_i\, \alpha_k(i)\, a_{ij}]\, b_j(o_{k+1})\,, \quad 1 <= j <= N,\ 1 <= k <= K-1.$$

- Termination:

$$P(o_1\, o_2 \ldots\, o_K) = \Sigma_i\, P(o_1\, o_2 \ldots\, o_K, q_K = s_i) = \Sigma_i\, \alpha_K(i)$$

- Complexity :

  N^2K operations.

Backward recursion for HMM

- Define the backward variable $\beta_k(i)$ as the joint probability of the partial observation sequence $o_{k+1}\, o_{k+2} \ldots\, o_K$ given that the hidden state at time k is s_i : $\beta_k(i)= P(o_{k+1}\, o_{k+2} \ldots\, o_K | q_k= s_i)$
- Initialization:

$$\beta_K(i)= 1 \quad , 1<=i<=N.$$

- Backward recursion:

$$\beta_k(j)= P(o_{k+1}\, o_{k+2} \ldots\, o_K | q_k= s_j) =$$
$$\Sigma_i\, P(o_{k+1}\, o_{k+2} \ldots\, o_K ,\, q_{k+1}= s_i | q_k= s_j) =$$
$$\Sigma_i\, P(o_{k+2}\, o_{k+3} \ldots\, o_K | q_{k+1}= s_i)\; a_{ji}\; b_i\, (o_{k+1}) =$$
$$\Sigma_i\, \beta_{k+1}(i)\; a_{ji}\; b_i\, (o_{k+1}) , \qquad 1<=j<=N,\ 1<=k<=K-1.$$

- Termination:

$$P(o_1\, o_2 \ldots\, o_K) = \Sigma_i\, P(o_1\, o_2 \ldots\, o_K ,\, q_1= s_i) =$$
$$\Sigma_i\, P(o_1\, o_2 \ldots\, o_K | q_1= s_i)\; P(q_1= s_i) = \Sigma_i\, \beta_1(i)\; b_i\, (o_1)\; \pi_i$$

Decoding problem

•**Decoding problem.** Given the HMM $M=(A, B, \pi)$ and the observation sequence $O=o_1 o_2 \ldots o_K$, calculate the most likely sequence of hidden states s_i that produced this observation sequence.

• We want to find the state sequence $Q= q_1 \ldots q_K$ which maximizes $P(Q \mid o_1 o_2 \ldots o_K)$, or equivalently $P(Q , o_1 o_2 \ldots o_K)$.

• Brute force consideration of all paths takes exponential time. Use efficient Viterbi algorithm instead.

• Define variable $\delta_k(i)$ as the maximum probability of producing observation sequence $o_1 o_2 \ldots o_k$ when moving along any hidden state sequence $q_1 \ldots q_{k-1}$ and getting into $q_k= s_i$.

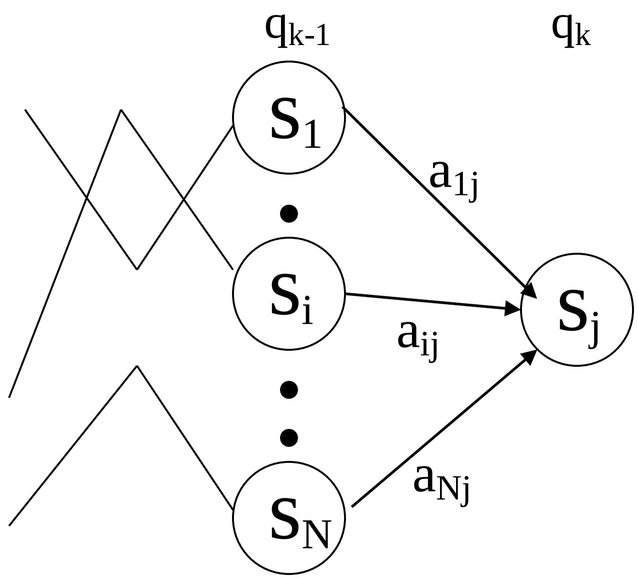
$$\delta_k(i) = \max P(q_1 \ldots q_{k-1} , q_k= s_i , o_1 o_2 \ldots o_k)$$

where max is taken over all possible paths $q_1 \ldots q_{k-1}$.

Viterbi algorithm (1)

- General idea:

  if best path ending in $q_k = s_j$ goes through $q_{k-1} = s_i$ then it should coincide with best path ending in $q_{k-1} = s_i$.



- $\delta_k(i) = \max P(q_1 \ldots q_{k-1}, q_k = s_j, o_1 o_2 \ldots o_k) =$
$\max_i [\, a_{ij}\, b_j(o_k) \max P(q_1 \ldots q_{k-1} = s_i, o_1 o_2 \ldots o_{k-1}) \,]$

- To backtrack best path keep info that predecessor of s_j was s_i.

Viterbi algorithm (2)

- Initialization:

$$\delta_1(i) = \max P(q_1 = s_i , o_1) = \pi_i \, b_i(o_1) , \quad 1<=i<=N.$$

- Forward recursion:

$$\delta_k(j) = \max P(q_1 \ldots q_{k-1} , q_k = s_j , o_1 o_2 \ldots o_k) =$$

$$\max_i [a_{ij} \, b_j(o_k) \max P(q_1 \ldots q_{k-1} = s_i , o_1 o_2 \ldots o_{k-1})] =$$

$$\max_i [a_{ij} \, b_j(o_k) \, \delta_{k-1}(i)] , \quad 1<=j<=N, 2<=k<=K.$$

- Termination: choose best path ending at time K

$$\max_i [\delta_K(i)]$$

- Backtrack best path.

This algorithm is similar to the forward recursion of evaluation problem, with Σ *replaced by max and additional backtracking.*

Learning problem (1)

• **Learning problem.** Given some training observation sequences $O = o_1 o_2 \ldots o_K$ and general structure of HMM (numbers of hidden and visible states), determine HMM parameters $M = (A, B, \pi)$ that best fit training data, that is maximizes $P(O \mid M)$.

• There is no algorithm producing optimal parameter values.

• Use iterative expectation-maximization algorithm to find local maximum of $P(O \mid M)$ (Baum-Welch algorithm).

Expectation Maximization (EM)

Iteratively finding maximum likelihood using partial observation.

X: observed data
Z: unobserved data: (latent)
θ: Model parameters

$$P(X|\theta) = \int P(X, Z|\theta)dz = \int P(X|Z, \theta)P(Z|\theta)dz$$

Expectation Maximization (EM)

E-Step (Expectation)

$Q\left(\theta\middle|\theta^{(t)}\right)$ =Expected latent log likelihood of θ

$$Q\left(\theta\middle|\theta^{(t)}\right) = E[L(\theta; X; Z)]$$

M-Step (Maximization)

$$\theta^{(t+1)} = argmax_{\theta}Q\left(\theta\middle|\theta^{(t)}\right)$$

Learning problem (2)

- If training data has information about sequence of hidden states (as in word recognition example), then use maximum likelihood estimation of parameters:

$$a_{ij} = P(s_i \mid s_j) = \frac{\text{Number of transitions from state } S_j \text{ to state } S_i}{\text{Number of transitions out of state } S_j}$$

$$b_i(v_m) = P(v_m \mid s_i) = \frac{\text{Number of times observation } V_m \text{ occurs in state } S_i}{\text{Number of times in state } S_i}$$

Baum-Welch algorithm

General idea:

$$a_{ij} = P(s_i \mid s_j) = \frac{\text{Expected number of transitions from state } S_j \text{ to state } S_i}{\text{Expected number of transitions out of state } S_j}$$

$$b_i(v_m) = P(v_m \mid s_i) = \frac{\text{Expected number of times observation } V_m \text{ occurs in state } S_i}{\text{Expected number of times in state } S_i}$$

$\pi_i = P(s_i) =$ Expected frequency in state S_i at time $k=1$.

Baum-Welch algorithm: expectation step(1)

- Define variable $\xi_k(i,j)$ as the probability of being in state s_i at time k and in state s_j at time k+1, given the observation sequence $o_1\, o_2 \ldots\, o_K$.

$$\xi_k(i,j) = P(q_k = s_i\, , q_{k+1} = s_j \mid o_1\, o_2 \ldots\, o_K)$$

$$\xi_k(i,j) = \frac{P(q_k = s_i\, , q_{k+1} = s_j\, , o_1\, o_2 \ldots\, o_k)}{P(o_1\, o_2 \ldots\, o_k)} =$$

$$\frac{P(q_k = s_i\, , o_1\, o_2 \ldots\, o_k)\; a_{ij}\; b_j(o_{k+1})\; P(o_{k+2} \ldots\, o_K \mid q_{k+1} = s_j)}{P(o_1\, o_2 \ldots\, o_k)} =$$

$$\frac{\alpha_k(i)\; a_{ij}\; b_j(o_{k+1})\; \beta_{k+1}(j)}{\Sigma_i \Sigma_j\; \alpha_k(i)\; a_{ij}\; b_j(o_{k+1})\; \beta_{k+1}(j)}$$

Baum-Welch algorithm: expectation step(2)

- Define variable $\gamma_k(i)$ as the probability of being in state s_i at time k, given the observation sequence $o_1\, o_2 \ldots\, o_K$.

$$\gamma_k(i) = P(q_k = s_i \mid o_1\, o_2 \ldots\, o_K)$$

$$\gamma_k(i) = \frac{P(q_k = s_i,\, o_1\, o_2 \ldots\, o_k)}{P(o_1\, o_2 \ldots\, o_k)} = \frac{\alpha_k(i)\, \beta_k(i)}{\Sigma_i\, \alpha_k(i)\, \beta_k(i)}$$

Baum-Welch algorithm: expectation step(3)

- We calculated $\xi_k(i,j) = P(q_k = s_i, q_{k+1} = s_j \mid o_1 o_2 \ldots o_K)$
  and $\gamma_k(i) = P(q_k = s_i \mid o_1 o_2 \ldots o_K)$

- Expected number of transitions from state s_i to state s_j =
  $= \sum_k \xi_k(i,j)$
- Expected number of transitions out of state $s_i = \sum_k \gamma_k(i)$

- Expected number of times observation v_m occurs in state s_i =
  $= \sum_k \gamma_k(i)$, k is such that $o_k = v_m$
- Expected frequency in state s_i at time $k=1$: $\gamma_1(i)$.

Baum-Welch algorithm: maximization step

$$a_{ij} = \frac{\text{Expected number of transitions from state } s_j \text{ to state } s_i}{\text{Expected number of transitions out of state } s_j} = \frac{\sum_k \xi_k(i,j)}{\sum_k \gamma_k(i)}$$

$$b_i(v_m) = \frac{\text{Expected number of times observation } v_m \text{ occurs in state } s_i}{\text{Expected number of times in state } s_i} = \frac{\sum_k \xi_k(i,j)}{\sum_{k, o_k = v_m} \gamma_k(i)}$$

$$\pi_i = (\text{Expected frequency in state } s_i \text{ at time } k=1) = \gamma_1(i).$$

Next Week:

Sampling

Have a good day!