# سوال اول

1. Determine the correctness or incorrectness of the following statements and provide reasoning for each :

   (a) $\{N_t\}$ (Poisson process) is a stationary random process.

   (b) It is possible to determine whether a random process is SSS by observing a sample path of the process.

   (c) Assume $N(t)$ is a Poisson process with rate $\lambda$ and $X_1$ is the time of the first arrival. Given that $N(t) = 1$, $X_1$ is uniformly distributed over the interval $(0, t]$.

(f) Consider the process $\{X(t); t \in \mathbb{R}\}$ where $X(t) = At$ and $A \sim N(0, 1)$. The covariance of $X(t)$ is given by $\text{Cov}(X(t_1), X(t_2)) = t_1 t_2$.

Solution:

(a) False: For example, $\mathbb{E}[N_t] = \lambda t$, which depends on $t$.

(b) False: According to the definitions, this statement is clearly incorrect.

(c) Correct: Suppose that for a Poisson process at rate $\lambda$, we condition on the event $\{N(t) = 1\}$, the event that exactly one arrival occurred during $(0, t]$. We might conjecture that under such conditioning, $t_1$ should be uniformly distributed over $(0, t)$. To see that this is in fact so, choose $s \in (0, t)$. Then

$$\mathbb{P}(t_1 \leq s \mid N(t) = 1) = \frac{\mathbb{P}(t_1 \leq s, N(t) = 1)}{\mathbb{P}(N(t) = 1)}$$

$$= \frac{\mathbb{P}(N(s) = 1, N(t) - N(s) = 0)}{\mathbb{P}(N(t) = 1)}$$

$$= \frac{e^{-\lambda s} \lambda s e^{-\lambda(t-s)}}{e^{-\lambda t} \lambda t}$$

$$= \frac{s}{t}.$$

(f) Correct: **Auto-Covariance Function:**

$$\text{Cov}(X(t_1), X(t_2)) = \mathbb{E}[X(t_1)X(t_2)] = \mathbb{E}[t_1 A \cdot t_2 A] = t_1 t_2 \mathbb{E}[A^2] = t_1 t_2 \cdot \text{Var}(A) = t_1 t_2.$$

سوال دوم

2. Joyce receives Facebook updates from her three closest friends: Amy, Brenda and Cindy. Each of her friends has different Facebook habits, with Amy sending her messages at a rate of 1/7 per day, Brenda at a rate of 1/2 per day, and Cindy at a rate of 1 per day. We assume that the number of messages Joyce receives from her friends are independent Poisson processes.

   (a) [5 PTS] Let $N(t)$ denote the number of messages Joyce receives from her three friends during the period $[0, t]$. Write down the PMF of $N(t)$, i.e., $P(N(t) = n)$ for $n = 0, 1, 2, \ldots$.

   *Solution:*

   Since the superposition of independent Poisson processes is again a Poisson process whose rate is the sum of the rates, then $N(t)$ is a Poisson process with rate

$$\frac{1}{7} + \frac{1}{2} + 1 = \frac{2 + 7 + 14}{14} = \frac{23}{14}$$

   or equivalently,

$$P(N(t) = n) = \frac{e^{-23t/14}(23t/14)^n}{n!}, \ldots n = 0, 1, 2, \ldots.$$

(b) [8 PTS] Compute the probability that the next Facebook update Joyce receives from her three friends comes from Cindy.

*Solution:*

Since this is the same as the probability that the minimum of three independent exponentials, $\chi_A$, $\chi_B$ and $\chi_C$, having rates 1/7, 1/2 and 1, respectively, is equal to $\xi_C$, then

$$P(\text{Next update is from Cindy}) = P(\chi_C = \min\{\chi_A, \chi_B, \chi_C) = \frac{1}{1/7 + 1/2 + 1} = \frac{14}{23}$$

(c) [8 PTS] Given that yesterday Joyce received 5 Facebook updates from her group of three friends (during a 24 hour period), compute the probability that none of them are from Cindy.

*Solution:*

Let $N_A(t)$, $N_B(t)$, and $N_C(t)$ denote the number of Facebook updates from Amy, Brenda and Cindy, respectively, that Joyce received during the period $[0, t]$. Then, we need to compute

$$\begin{aligned}
P(N_C(t) = 0 | N(t) = 5) &= \frac{P(N_C(t) = 0,\ N(t) = 5)}{P(N(t) = 5)} \\
&= \frac{P(N_C(t) = 0, N_A(t) + N_B(t) + N_C(t) = 5)}{P(N(t) = 5)} \\
&= \frac{P(N_C(t) = 0)P(N_A(t) + N_B(t) = 5)}{P(N(t) = 5)} \qquad \text{(by independence)} \\
&= \frac{e^{-1} \cdot e^{-(1/7+1/2)}(1/7 + 1/2)^5/5!}{e^{-(23/14)}(23/14)^5/5!} \\
&= \frac{(1/7 + 1/2)^5}{(23/14)^5} = \left(\frac{9}{23}\right)^5
\end{aligned}$$

# سوال سوم

## Question

Let $\{X_t\}$ be a weakly stationary process with mean zero and autocovariance function $\gamma_X(\tau)$. Define a new process $Z_t$ by:

$$Z_t = X_t X_{t-1}. \tag{1}$$

1. Determine whether $\{Z_t\}$ is weakly stationary by calculating its mean and autocovariance function.
2. If $X_t$ is a white noise process with variance $\sigma^2$, determine if $\{Z_t\}$ is weakly stationary. If so, calculate its mean and autocovariance function.

## Answer

### 1. Determine if $Z_t = X_t X_{t-1}$ is weakly stationary

**Mean of $Z_t$:** Since $X_t$ is weakly stationary with mean zero, we have:

$$\mathbb{E}[Z_t] = \mathbb{E}[X_t X_{t-1}] = \gamma_X(1), \tag{2}$$

where $\gamma_X(1)$ is the autocovariance at lag 1. Therefore, the mean of $Z_t$ is constant over time, assuming $\gamma_X(1)$ itself is constant.

**Autocovariance of $Z_t$:** To check weak stationarity, we need to compute $\text{Cov}(Z_t, Z_{t+\tau})$ and see if it depends only on $\tau$ and not on $t$. By expanding $Z_t = X_t X_{t-1}$ and $Z_{t+\tau} = X_{t+\tau} X_{t+\tau-1}$, we get:

$$\text{Cov}(Z_t, Z_{t+\tau}) = \mathbb{E}[(X_t X_{t-1})(X_{t+\tau} X_{t+\tau-1})] - \mathbb{E}[X_t X_{t-1}]\mathbb{E}[X_{t+\tau} X_{t+\tau-1}]. \tag{3}$$

Since $\{X_t\}$ is weakly stationary, each term $\mathbb{E}[X_t X_{t-1}] = \gamma_X(1)$, so we simplify:

$$\text{Cov}(Z_t, Z_{t+\tau}) = \mathbb{E}[X_t X_{t-1} X_{t+\tau} X_{t+\tau-1}] - \gamma_X(1)^2. \tag{4}$$

To proceed further, we would need higher-order moments of $X_t$, which depend on the specific distribution of $X_t$. Thus, without additional assumptions on $X_t$, we cannot conclude whether $Z_t$ is weakly stationary.

### 2. If $X_t$ is a white noise process with variance $\sigma^2$:

**Mean of $Z_t$:** If $X_t$ is white noise, then $\mathbb{E}[X_t X_{t-1}] = 0$ since white noise is uncorrelated across time. Therefore:

$$\mathbb{E}[Z_t] = \mathbb{E}[X_t X_{t-1}] = 0. \tag{5}$$

**Autocovariance of $Z_t$:** For white noise, $X_t$ and $X_{t-1}$ are independent, so we can expand $\mathbb{E}[X_t X_{t-1} X_{t+\tau} X_{t+\tau-1}]$ based on whether $\tau = 0, 1$, or $\tau \geq 2$:

- If $\tau = 0$:
$$\text{Cov}(Z_t, Z_t) = \mathbb{E}[(X_t X_{t-1})^2] = \mathbb{E}[X_t^2]\mathbb{E}[X_{t-1}^2] = \sigma^4. \tag{6}$$
- If $\tau = 1$:
$$\text{Cov}(Z_t, Z_{t+1}) = \mathbb{E}[X_t X_{t-1} X_{t+1} X_t] = \mathbb{E}[X_t^2]\mathbb{E}[X_{t-1} X_{t+1}] = 0, \tag{7}$$
since $X_{t-1}$ and $X_{t+1}$ are uncorrelated in a white noise process.
- If $\tau \geq 2$:
$$\text{Cov}(Z_t, Z_{t+\tau}) = \mathbb{E}[X_t X_{t-1} X_{t+\tau} X_{t+\tau-1}] = \mathbb{E}[X_t X_{t-1}]\mathbb{E}[X_{t+\tau} X_{t+\tau-1}] = 0. \tag{8}$$

Therefore, the autocovariance function $\gamma_Z(\tau)$ for $\{Z_t\}$ is:

$$\gamma_Z(\tau) = \begin{cases} \sigma^4 & \text{if } \tau = 0, \\ 0 & \text{if } \tau \neq 0. \end{cases} \tag{9}$$

This implies that $Z_t$ is weakly stationary, with mean zero and an autocovariance structure similar to white noise, as all covariances $\gamma_Z(\tau)$ vanish for $\tau \neq 0$.

## سوال چهارم

3. Consider a WSS random sequence $X[n]$ with mean function $\mu_X$, a constant, and correlation function $R_{XX}[m]$. Form a random process as

$$X(t) = \sum_{n=-\infty}^{\infty} X[n] \frac{sin\pi(t-nT)/T}{\pi(t-nT)/T}$$

(a) Find $\mu_X(t)$ in terms of $\mu_X$.

(b) Find $R_{XX}(t_1, t_2)$ in terms of $R_{XX}[m]$. Is $X(t)$ WSS?

Solution:

Consider a WSS (Wide-Sense Stationary) random sequence $X[n]$ with: - Mean function $\mu_X$ (a constant) - Correlation function $R_{XX}[m]$

The random process $X(t)$ is defined as:

$$X(t)=\sum_{n=-\infty}^{\infty} X[n]\frac{\sin\pi(t-nT)/T}{\pi(t-nT)/T}$$

Part (a) Find $\mu_X(t)$ in terms of $\mu_X$.

The mean of the process $X(t)$, denoted as $\mu_X(t)=E[X(t)]$, is given by:

$$\mu_X(t)=E\left[\sum_{n=-\infty}^{\infty} X[n]\frac{\sin\pi(t-nT)/T}{\pi(t-nT)/T}\right]$$

Since expectation is a linear operator:

$$\mu_X(t)=\sum_{n=-\infty}^{\infty} E[X[n]]\frac{\sin\pi(t-nT)/T}{\pi(t-nT)/T}$$

Given $E[X[n]]=\mu_X$ for all $n$ (because $X[n]$ is WSS):

$$\mu_X(t)=\mu_X\sum_{n=-\infty}^{\infty}\frac{\sin\pi(t-nT)/T}{\pi(t-nT)/T}$$

The sum:

$$\sum_{n=-\infty}^{\infty}\frac{\sin\pi(t-nT)/T}{\pi(t-nT)/T}=1$$

This is due to the property of the Dirichlet kernel (sinc function as a sampling reconstruction kernel).

Thus:

$$\mu_X(t)=\mu_X$$

Part (b) Find $R_{XX}(t_1,t_2)$ in terms of $R_{XX}[m]$. Is $X(t)$ WSS?

The autocorrelation function $R_{XX}(t_1,t_2)$ is defined as:

$$R_{XX}(t_1,t_2)=E[X(t_1)X(t_2)]$$

Substituting the definition of $X(t)$:

$$R_{XX}(t_1,t_2)=E\left[\left(\sum_{n=-\infty}^{\infty} X[n]\frac{\sin\pi(t_1-nT)/T}{\pi(t_1-nT)/T}\right)\left(\sum_{m=-\infty}^{\infty} X[m]\frac{\sin\pi(t_2-mT)/T}{\pi(t_2-mT)/T}\right)\right]$$

Expanding the expectation:

$$R_{XX}(t_1,t_2)=\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{\sin\pi(t_1-nT)/T}{\pi(t_1-nT)/T}\frac{\sin\pi(t_2-mT)/T}{\pi(t_2-mT)/T}E[X[n]X[m]]$$

Since $X[n]$ is WSS, $E[X[n]X[m]] = R_{XX}[n-m]$:

$$R_{XX}(t_1,t_2) = \sum_{k=-\infty}^{\infty} R_{XX}[k] \sum_{n=-\infty}^{\infty} \frac{\sin \pi(t_1 - nT)/T}{\pi(t_1 - nT)/T} \frac{\sin \pi(t_2 - (n-k)T)/T}{\pi(t_2 - (n-k)T)/T}$$

Using the orthogonality of the sinc functions:

$$R_{XX}(t_1,t_2) = \sum_{k=-\infty}^{\infty} R_{XX}[k] \frac{\sin \pi(t_1 - t_2 - kT)/T}{\pi(t_1 - t_2 - kT)/T}$$

Now, for $X(t)$ to be WSS, $R_{XX}(t_1,t_2)$ should depend only on $\tau = t_1 - t_2$:

$$R_{XX}(t_1,t_2) = R_{XX}(\tau)$$

Thus, $X(t)$ is WSS if $R_{XX}(t_1,t_2)$ only depends on $\tau$, which is satisfied here since it depends on $\tau$ through $t_1 - t_2$.

Conclusion 1. $\mu_X(t) = \mu_X$ 2. $R_{XX}(t_1,t_2)$ is derived in terms of $R_{XX}[m]$, and $X(t)$ is WSS.

# سوال پنجم

فرض کنید که $X(t)$ یک فرایند گوسی با میانگین صفر و $R_X(\tau) = sinc(\tau)$ باشد. اگر $X(t)$ ورودی سیستم LTI زیر باشد:

$$H(f) = \begin{cases} 1 & |f| < 1 \\ 0 & \text{otherwise} \end{cases} \quad (۱)$$

و $Y(t)$ خروجی آن باشد. $P(Y(2) < 1|Y(1) < 1)$ را محاسبه کنید.

جواب:
با توجه به اینکه ورودی یک فرایند گوسی wss است و از یک سیستم LTI عبور داده شده پس خروجی هم یک فرایند گوسی WSS است. چون میانگین ورودی صفر است پس در خروجی هم میانگین صفر است. برای محاسبه ی $R_Y(\tau)$ داریم:

$$S_X(f) = \begin{cases} 1 & |f| < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \quad (۲)$$

$$S_Y(f) = S_X(f)|H(f)|^2 = \begin{cases} 1 & |f| < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \quad (۳)$$

در نتیجه: $R_Y(\tau) = sinc(\tau)$
بنابراین: $E[Y(1)^2] = sinc(0) = 1 \Rightarrow Var(Y(1) = 1)$
از طرفی داریم: $E[Y(1)Y(2)] = R_Y(1) = 0$ و میدانیم توزیع مشترک $Y(1), Y(2)$ گوسی است و چون کورلیشن این دو (توزیع نرمال)باهم صفر است دو توزیع مستقل از هم هستند. و داریم:
$P(Y(2) < 1|Y(1) < 1) = P(Y(2) < 1) = \phi(1)$ که در آن $\phi()$ تابع استانداردCDF است.