Sharif University of Technology

Stochastic Process

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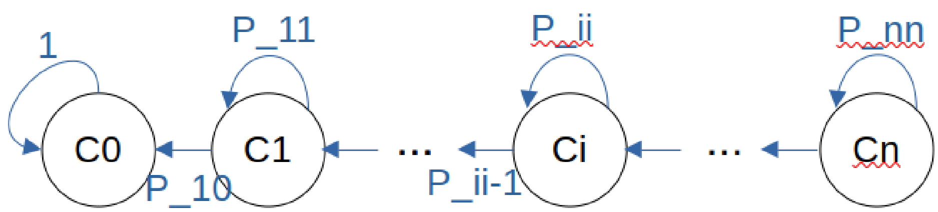
Quiz 9 (25) | Markov Chains and Hidden Markov Models | Release: Dey, 4th

1. Jimmy has a box of N toys, some clean and some dirty. Each day, he randomly picks up a toy, plays with it and puts it back in the box. Whichever toy he plays, turns dirty at the end of the day with a probability of q.

   - [2] Model the status of the box as a Markov Chain, assuming the types of the toys is not important in general status calculation. Why does this model satisfy the Markov property?
   - [6] Find the transition probabilities.
   - [2] List the transient and recurrent states.

   Solution:

   - The number of clean toys each day, B_d, completely specifies the status of the toy box. Let the states be denoted as $c_0, c_1, \cdots, c_i, \cdots, c_n$, in which c_i stands for the existence of i clean toys in the box. We can have the following Markov Chain:



     As the number of clean toys each day, B_d, specifies the probability distribution function of the number of clean toys on the next day, B_{d+1}, without requiring the history; the process has the Markov Property.

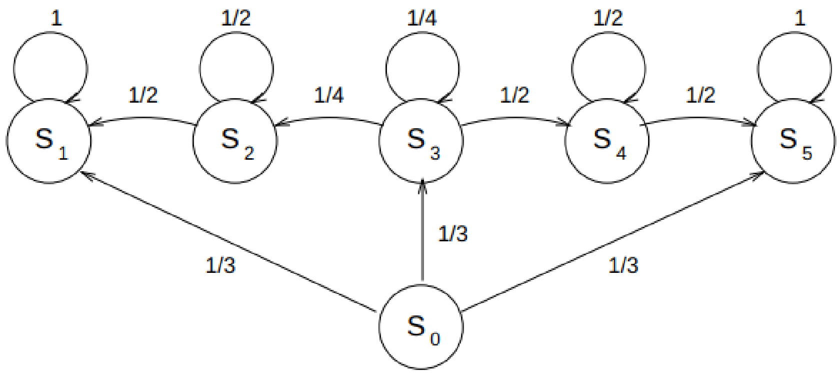
$$P(B_{d+1} = c_i | B_0 = c_{j_0}, B_1 = cj_1, \cdots, B_d = c_{j_d}) = P(B_{d+1} = c_i | B_d = c_{j_d}) \tag{1}$$

   - Assume having c_i clean toys in the box. The probability of choosing a clean toy for playing would be $\frac{c_i}{N}$, and a dirty toy would be $\frac{N-c_i}{N}$. If we choose a dirty toy, the state won't be changed. But if we choose a clean one, the state would change with a probability of q:

$$\begin{cases} P_{i,i} = \dfrac{N - c_i}{N} + (\dfrac{c_i}{N}) * (1 - q) \\ P_{i,i-1} = \dfrac{c_i}{N} * q \end{cases} \tag{2}$$

   - c_0 is the only recurrent state, as it is an absorbing state. All of the others are transient.

2. Consider the following Markov chain:



Assume starting from state s_0, answer the following questions:

- [3] Entering the state s_2 for the first time as the result of Kth trial.
- [4] Never entering s_4.
- [3] Entering s_2 and leaving it on the text trial.

Solution:

- Assume that A_k specifies the state of the system at trial k. The only way to enter state s_2 for the first time on the kth trial is to enter state s_3 on the first trial, remain in s_3 for the next k − 2 trials, and finally enter s_2 on the last trial. Thus:

$$P(A_k = s_2) = p_{03} \times p_{33}^{k-2} \times p_{32} = \frac{1}{3} \times (\frac{1}{4})^{k-1} \tag{3}$$

- There are 3 possible ways for this scenario:
  (a) $s_0 \rightarrow s_1 = p_{01} = \frac{1}{3}$
  (b) $s_0 \rightarrow s_5 = p_{05} = \frac{1}{3}$
  (c) $s_0 \rightarrow s_3 \rightarrow \cdots \rightarrow s_2 = \frac{1}{3} \times (\Sigma_{i=0}^{\infty}(\frac{1}{4})^i) \times \frac{1}{4} = \frac{1}{3} \times \frac{4}{3} \times \frac{1}{4} = \frac{1}{9}$
  Thus the total probability is $\frac{2}{3} + \frac{1}{9} = \frac{7}{9}$
- Considering the first bullet, the answer is: $\Sigma_{k=2}^{\infty} P(A_k = s_2) \times P(A_{k+1} \neq s_2)$
  $= (\Sigma_{k=2}^{\infty} \frac{1}{3} \times (\frac{1}{4})^{k-1}) \times \frac{1}{2} = \frac{1}{6} \times \frac{\frac{1}{4}}{1-\frac{1}{4}} = \frac{1}{18}$