In the name of GOD.

Sharif University of Technology

Stochastic Process

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Quiz 3 (25 minutes) | Ergodicity + Power Spectrum | Release : Oct. 20th

1. Determine whether a random process with the following autocorrelation functions can be ergodic. (20 points)

   - $R_{XX}(t, s) = e^{-|ts|}$
     Solution: Remember that

     $$\text{w.s.s. processes} \supset \text{s.s.s. processes} \supset \text{ergodic processes.}$$

     An ergodic process is necessarily strict-sense stationary, which requires the autocorrelation to be a function of the time difference, $\tau = t - s$, and not the specific time indices. Since this autocorrelation depends on both t and s, the process cannot be ergodic.

   - $R_{XX}(\tau)$ with $R_{XX}(0) = 3$ and $R_{XX}(5) = 10$
     Solution: An ergodic process must be wide-sense stationary, which implies $R_{XX}(0) \geq |R_{XX}(\tau)|$ for all τ. Since $R_{XX}(5) > R_{XX}(0)$, this condition is violated, so the process cannot be ergodic.

2. Let $X(t)$ be a wide-sense stationary process with autocorrelation function $R_{XX}(\tau) = ce^{-|\tau|}$ and a total power of 4. What is its power spectral density $S_{XX}(\omega)$? (30 points)

   Hint: $\mathcal{F}\{e^{-a|\tau|}\} = \frac{2a}{a^2+\omega^2}$

   Solution: The total power of the process is given by:

   $$P = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XX}(\omega)\, d\omega = R_{XX}(0).$$

   Since the total power is 4, we have:

   $$R_{XX}(0) = c = 4.$$

   The power spectral density is the Fourier transform of the autocorrelation function:

   $$\begin{aligned} S_{XX}(\omega) &= \mathcal{F}\{R_{XX}(\tau)\} = \mathcal{F}\{4e^{-|\tau|}\} \\ &= 4\mathcal{F}\{e^{-|\tau|}\} = \frac{8}{1+\omega^2}. \end{aligned}$$

3. Check if a random process $X(t)$ with $E[X(t)] = 0$ and $R_{XX}(s, t) = 9 + 4e^{-0.2|s-t|}$ is mean-ergodic. (50 points)

   Solution: For a process to be mean-ergodic, we need to check if:

   $$E\left[\left(\frac{1}{2T}\int_{-T}^{T} (X(t) - \mu_X)\, dt\right)^2\right]$$

converges to 0 as $T \to \infty$. Since $E[X(t)] = 0$, this simplifies to:

$$\begin{aligned}
E\left[\left(\frac{1}{2T}\int_{-T}^{T} X(t)\,dt\right)^2\right] &= \frac{1}{4T^2}\int_{-T}^{T}\int_{-T}^{T} E[X(t)X(s)]\,dt\,ds \\
&= \frac{1}{4T^2}\int_{-T}^{T}\int_{-T}^{T} R_{XX}(t,s)\,dt\,ds \\
&= \frac{1}{4T^2}\int_{-T}^{T}\int_{-T}^{T} \left(9 + 4e^{-0.2|t-s|}\right)\,dt\,ds \\
&\geq \frac{1}{4T^2}\int_{-T}^{T}\int_{-T}^{T} 9\,dt\,ds \\
&= \frac{9}{4},
\end{aligned}$$

which does not converge to 0. Hence, the process is not mean-ergodic.