Time: 15 mins

Name: **Std. Number:**

Quiz 2

1. Consider the process

$$X_t = \alpha X_{t-1} + \epsilon_t, \quad \epsilon_t \sim N(\mu, \sigma^2),\ 0 < \alpha < 1,\ X_0 = 0,$$

where ϵ_t are independent and identically distributed (i.i.d.).

(a) Find $\mathbb{E}(X_t)$.

(b) Calculate $\text{Var}(X_t)$.

(c) Determine the autocorrelation function.

(d) Assess if the process is Wide-Sense Stationary (W.S.S) for $t > T$, where T is a sufficiently large integer.

Solution:

1. Expected Value $E(X_t)$
We know $E(X_t) = E(\alpha X_{t-1} + \epsilon_t) = \alpha E(X_{t-1}) + \mu$. The recurrence relation is:

$$\mu_X(t) = \alpha \mu_X(t-1) + \mu, \quad \mu_X(0) = 0.$$

Solving the recurrence using the geometric series gives:

$$\mu_X(t) = \frac{\mu(1-\alpha^t)}{1-\alpha}.$$

2. Variance Var(X_t)
For variance, using the law of total variance:

$$\text{Var}(X_t) = \sigma^2 + \alpha^2 \text{Var}(X_{t-1}).$$

Solving this recurrence relation gives:

$$\text{Var}(X_t) = \sigma^2 \cdot \frac{1-\alpha^{2t}}{1-\alpha^2}.$$

3. Autocorrelation Function
The autocorrelation is:

$$R_X(t, t+k) = E[X_t X_{t+k}] = \alpha^k \cdot \mathrm{Var}(X_t) = \alpha^k \cdot \frac{\sigma^2(1-\alpha^{2t})}{1-\alpha^2}.$$

4. Wide-Sense Stationarity (W.S.S)
A process is W.S.S if:

- $E[X_t]$ is constant over time.
- $R_X(t, t+k)$ depends only on lag k.

For $E[X_t]$:

$$\lim_{t\to\infty} E[X_t] = \frac{\mu}{1-\alpha}.$$

For $R_X(t, t+k)$, as $t \to \infty$:

$$R_X(t, t+k) \to \alpha^k \cdot \frac{\sigma^2}{1-\alpha^2},$$

which depends only on k. Hence, the process is W.S.S for $t > T$.