In the name of GOD.

Stochastic Process

Fall 2024

Sharif University of Technology

Hamid R. Rabiee

Homework 5 | Hypothesis Testing, Markov Chains, Hidden Markov Model | Deadline : 1403/10/01

1. [40] In automated food packaging machines, there is always a degree of error. We intend to purchase and evaluate the performance of several packaging machines for a food company.

   (a) There are two machines available for purchase at the same price. Based on their reported performance specifications, determine which machine has better accuracy. Provide a detailed explanation for your choice.

   - Machine Type A : Has 98% accuracy within a range of $\pm 2\%$.
     (This means that if the packaging weight is set to 100 grams, 98% of the packages fall within the range of 98 to 102 grams.)
   - Machine Type B : Has 90% accuracy within a range of $\pm 1\%$.
     (This means that if the packaging weight is set to 100 grams, 90% of the packages fall within the range of 99 to 101 grams.)

   (b) After purchasing the chosen machine, we set it up and configured the target packaging weight to 100 grams. We measured the weights of 10 samples packaged by the machine, which are as follows:

   $$106.69, 107.1, 106.74, 106.23, 106.99, 107.72, 106.29, 108.51, 106.59, 106.76$$

   It seems the machine is not packaging correctly. Calculate the P-Value to determine if the machine is functioning correctly. ($\alpha = 0.05$)

   (c) After reviewing the machine's user manual, we discovered that the calibration of the machine degrades over time or due to transportation, introducing bias into its operation. Therefore, it needs to be recalibrated after transportation or at regular monthly intervals. The calibration process is iterative, reducing some bias with each step. However, due to the time-consuming nature of calibration, we decided to repeat it only a few steps. After each calibration step, 10 samples were taken to check the calibration accuracy of the machine. The samples taken at the end of each step are shown in Table 1. Based on this data, determine how many calibration steps are required. ($\alpha = 0.05$)

   (d) After years of use, these machines become worn, leading to operational drift. However, due to financial constraints, the company does not plan to replace them. As a result, we must continue the calibration process as before. Based on Table 2, determine how many calibration steps are required. ($\alpha = 0.05$) Have we achieved reliable calibration in the end?

Solution:

(a) The errors occurring in this scenario result from the interaction of many small errors in different parts of the system. Therefore, they can be modeled using a normal distribution.

Step 1	104.77	105.08	105.35	104.89	104.71	104.47	105.56	106.35	105.65	104.57
Step 2	103.98	102.67	103.63	102.91	103.01	103.95	104.36	103.56	103.53	103.72
Step 3	101.45	101.88	103.18	102.49	103.41	102.19	102.35	101.83	102.79	101.43
Step 4	102.57	101.56	101.66	102.07	102.19	101.86	100.81	101.71	100.74	101.87
Step 5	101.02	101.52	102.33	100.74	100.17	100.65	100.78	100.14	101.08	101.13
Step 6	101.01	100.04	101.58	100.75	100.44	101.04	100.31	100.66	100.39	100.49
Step 7	100.24	100.43	100.26	100.32	100.29	100.35	101.55	100.33	100.56	99.59
Step 8	100.19	101.07	101.18	100.32	100.19	101.11	99.16	101.03	99.53	99.73

Table 1: Sample data after each calibration step during setup

Step 1	91.29	92.64	97.74	95.75	95.71	92.11	98.63	92.13	91.68	91.73
Step 2	94.32	103.45	99.69	97.15	103.12	93.33	108.28	95.24	100.38	92.41
Step 3	99.68	96.11	98.93	103.91	103.2	101.01	95.28	103.29	98.58	94.98
Step 4	95.38	98.92	98.91	103.22	102.2	100.77	96.22	99.41	98.58	97.33
Step 5	107.45	102.52	89.22	98.19	96.73	90.65	101.74	104.21	106.00	98.61
Step 6	99.31	100.88	105.07	102.94	100.91	98.94	98.81	103.21	108.06	98.78

Table 2: Sample data from the calibration process of a worn machine

Consequently, it is necessary to calculate the variance of the distributions for two different systems and select the system with the lower variance.

- System Type A
  If the variance for this system is denoted as σ_A^2, then:

$$\frac{x-\mu}{\sigma_A} \sim \mathcal{N}(0,1)$$

where x represents the weight of the samples, and μ is the weight chosen for packaging. Based on the description of this system, we have:

$$P(0.98\mu \leq x \leq 1.02\mu) = 0.98$$

$$P\left(\frac{0.98\mu-\mu}{\sigma_A} \leq \frac{x-\mu}{\sigma_A} \leq \frac{1.02\mu-\mu}{\sigma_A}\right) = 0.98$$

$$P\left(\frac{-0.02\mu}{\sigma_A} \leq \frac{x-\mu}{\sigma_A} \leq \frac{0.02\mu}{\sigma_A}\right) = 0.98$$

Due to symmetry, we have:

$$\frac{-0.02\mu}{\sigma_A} = \Phi^{-1}\left(\frac{1-0.98}{2}\right) = \Phi^{-1}(0.01)$$

$$\sigma_A = \frac{-0.02\mu}{\Phi^{-1}(0.01)}$$

Referring to the z-table, we find:

$$\sigma_A = \frac{-0.02\mu}{-2.33} = 0.0086\mu$$

- System Type B

  Similar to the previous case, we can write:

$$\sigma_B = \frac{-0.01\mu}{\Phi^{-1}\left(\frac{1-0.90}{2}\right)} = \frac{-0.01\mu}{\Phi^{-1}(0.05)}$$

$$\sigma_B = \frac{-0.01\mu}{-1.64} = 0.0061\mu$$

Thus, the variance of system B is smaller, and it has higher accuracy.

(b) If the distribution of x is given as follows, then for $\bar{x}$, we have:

$$x \sim \mathcal{N}(\mu, \sigma_B)$$

$$\bar{x} \sim \mathcal{N}\left(\mu, \frac{\sigma_B}{\sqrt{n}}\right)$$

$$\bar{x} = \frac{106.69 + 107.1 + 106.74 + 106.23 + 106.99 + 107.72 + 106.29 + 108.51 + 106.59 + 106.76}{10} = 106.96$$

$$\mathcal{P}(|\bar{x} - \mu| > (106.96 - 100))$$

$$\Rightarrow P\text{-Value} = 2(1 - \phi\left(\frac{106.96 - 100}{\frac{\sigma_B}{\sqrt{n}}}\right)) = 2(1 - \phi(36.08)) = 2(1 - 1) = 0$$

As shown, the p-value is approximately zero, indicating that this system is not functioning correctly.

(c) Here, based on the explanations, we understand that the system's variance has not changed, and only the mean has experienced a slight bias. Therefore, we can still use σ_B for this calculation. Since the variance is known, the appropriate test here is the z-test.

Given that the variance remains consistent throughout all steps, we calculate the acceptable range for the mean at each step and then evaluate the steps.

$$Z = \frac{\bar{x} - \mu}{\frac{\sigma_B}{\sqrt{n}}}$$

The significance level is $\alpha = 0.05$, so for a two-tailed z-test, the value of $|Z|$ must be less than $|\Phi^{-1}(0.025)| = 1.96$. Therefore, the permissible range for $\bar{x}$ is:

$$\left|\frac{\bar{x} - \mu}{\frac{\sigma_B}{\sqrt{n}}}\right| < 1.96$$

$$\Rightarrow |\bar{x} - 100| < \frac{1.96 \cdot 0.0061 \cdot 100}{\sqrt{10}} = 0.378$$

$$\Rightarrow 100 - 0.378 < \bar{x} < 100 + 0.378$$

Now, we calculate the mean for each step.

As observed in table 3, 7 steps are sufficient, and step 8 can be omitted.

Step	$\bar{x}$
Step 1	105.14
Step 2	103.53
Step 3	102.30
Step 4	101.70
Step 5	100.96
Step 6	100.67
Step 7	100.36
Step 8	100.35

Table 3: Mean of different calibration steps during initialization

(d) Based on the explanations, we understand that the variance of the system has changed over time. Therefore, we can no longer use the z-test and must instead use the t-test.

$$t = \frac{\bar{x} - \mu}{\frac{\hat{\sigma}}{\sqrt{n}}}$$

$$\hat{\sigma} = \sqrt{\frac{1}{n-1} \sum (x_i - \bar{x})^2}$$

Since the number of data points is 10, the degrees of freedom will be 9 ($df = 9$). Based on the t-test table, for a two-tailed test, the value of $|t|$ must be less than 2.26. Now, we calculate the value of t for each step.

Step	t
Step 1	-6.96
Step 2	-0.77
Step 3	-0.47
Step 4	-1.15
Step 5	-0.24
Step 6	1.71

Table 4: Values of t during different calibration steps for the worn system

From Table 4, we observe that two steps are sufficient for calibration. The reason calibration completed earlier in this case is that the variance has increased, reducing the packaging accuracy.

2. [10] In a complete graph with n vertices ($n > 1$), consider two distinct vertices u and v.

(a) In a simple random walk, what is the expected number of steps to reach vertex v starting from vertex u?

(b) If a new vertex v' is added to the graph and connected to v with a single edge, what is the expected number of steps to reach vertex v' starting from vertex u?

Solution:

(a) We assume the expected number of steps for vertex v is 0, and for the other vertices, it is x. Thus, we have:

$$x = 1 + \left(\frac{n-2}{n-1}x + \frac{1}{n-1} \cdot 0\right)$$

$$x = 1 + \frac{n-2}{n-1}x$$

$$\frac{1}{n-1}x = 1$$

$$x = n - 1$$

(b) We assume the expected number of steps for vertex v' is 0, for vertex v is y, and for the remaining vertices is x. Thus, we have:

$$x = 1 + \left(\frac{n-2}{n-1}x + \frac{1}{n-1}y\right)$$

$$y = 1 + \left(\frac{n-1}{n}x + \frac{1}{n} \cdot 0\right) = 1 + \frac{n-1}{n}x$$

For x, we can write:

$$x = 1 + \left(\frac{n-2}{n-1}x + \frac{1}{n-1}\left(1 + \frac{n-1}{n}x\right)\right)$$

$$x = 1 + \frac{n-2}{n-1}x + \frac{1}{n-1} + \frac{1}{n}x$$

$$x - \frac{n-2}{n-1}x - \frac{1}{n}x = 1 + \frac{1}{n-1}$$

$$\frac{1}{n(n-1)}x = \frac{n}{n-1}$$

$$x = n^2$$

3. [10] A fair coin is flipped repeatedly until the sequence $\{Heads, Heads, Tails\}$ is observed. At this point, the experiment ends.

(a) Model the problem using a Markov chain.

(b) Calculate the expected number of flips until the experiment ends.

Solution:

(a) The Markov chain corresponding to this problem is shown in Figure 3a.

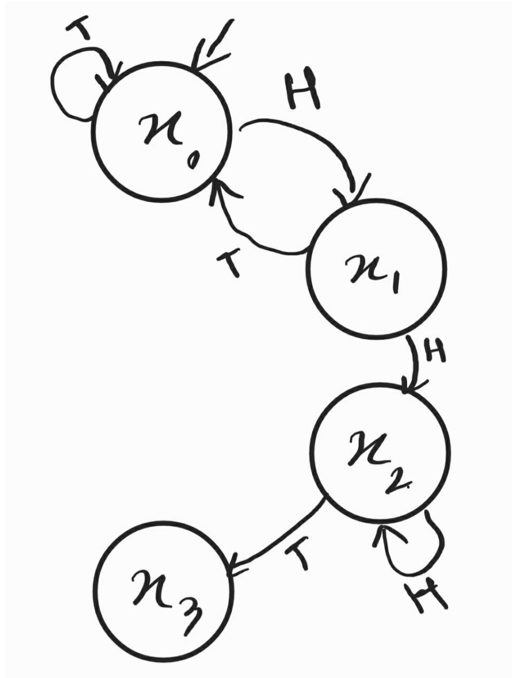

Figure 1: Markov chain of the coin-tossing problem

(b) If e_i represents the expected number of tosses until the end of the experiment starting from state x_i, we have:

$$e_3 = 0$$

$$e_2 = 1 + \left(\frac{1}{2}e_2 + \frac{1}{2}e_3\right)$$
$$\Rightarrow e_2 = 2$$

$$e_1 = 1 + \left(\frac{1}{2}e_2 + \frac{1}{2}e_0\right)$$
$$\Rightarrow e_1 = 1 + \left(\frac{1}{2} \cdot 2 + \frac{1}{2}e_0\right)$$
$$\Rightarrow e_1 = 2 + \frac{1}{2}e_0$$

For e_0, we have:

$$e_0 = 1 + \left(\frac{1}{2}e_0 + \frac{1}{2}e_1\right)$$

$$\Rightarrow e_0 = 1 + \left(\frac{1}{2}e_0 + \frac{1}{2}\left(2 + \frac{1}{2}e_0\right)\right)$$

$$\Rightarrow e_0 = 2 + \frac{3}{4}e_0$$

$$\Rightarrow e_0 = 8$$

$$\Rightarrow e_1 = 6$$

Thus, the expected number of coin tosses is 8.

4. [20] A tourist alternates between traveling for k_n days and resting for k_m days. If $k_n \sim \text{uniform}(1, n)$ and $k_m \sim \text{uniform}(1, m)$:

   (a) Model the state of this tourist as a Markov chain.

   (b) Using the model obtained in the previous part, determine the long-term probability that the tourist is traveling on the i-th day of their trip, given that $n_0 \leq i$ ($n_0 \leq n$).

Solution:

(a) In Figure 4a, you can see the Markov chain model for the tourist's trips. In this model, x_1 to x_n correspond to travel days, and x_{n+1} to x_{n+m} correspond to rest days.

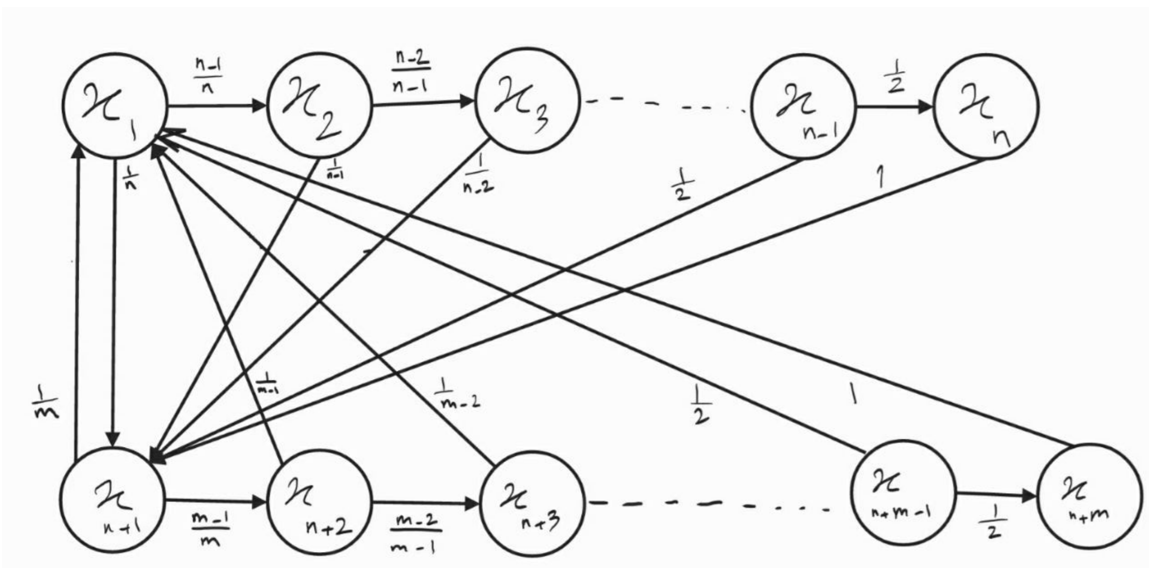

Figure 2: Markov process for the tourist's trips

(b) First, we need to determine the stationary distribution of this model. If π_i represents the probability of state x_i in the stationary distribution, we have:

$$\pi_2 = \frac{n-1}{n}\pi_1$$

$$\pi_3 = \frac{n-2}{n-1}\pi_2 = \frac{n-2}{n}\pi_1$$

$$\pi_4 = \frac{n-3}{n-2}\pi_3 = \frac{n-3}{n}\pi_1$$

$$(i \leq n) \Rightarrow \pi_i = \frac{n-i+1}{n}\pi_1$$

Similarly, for π_{n+2} to π_{n+m}:

$$\pi_{n+2} = \frac{m-1}{m}\pi_{n+1}$$

$$\pi_{n+3} = \frac{m-2}{m-1}\pi_{n+2} = \frac{m-2}{m}\pi_{n+1}$$

$$(i \leq m) \Rightarrow \pi_{n+i} = \frac{m-i+1}{m}\pi_{n+1}$$

For π_1, we have:

$$\pi_1 = \frac{1}{m}\pi_{n+1} + \frac{1}{m-1}\pi_{n+2} + \ldots + \frac{1}{1}\pi_{n+m}$$

$$= \frac{1}{m}\pi_{n+1} + \frac{1}{m-1}\frac{m-1}{m}\pi_{n+1} + \frac{1}{m-2}\frac{m-2}{m}\pi_{n+1} + \ldots + \frac{1}{1}\frac{1}{m}\pi_{n+1}$$

$$= m\frac{1}{m}\pi_{n+1} = \pi_{n+1}$$

We know $\sum_i \pi_i = 1$, so:

$$\sum_{i=1}^{n} \frac{n-i+1}{n}\pi_1 + \sum_{i=1}^{m} \frac{m-i+1}{m}\pi_{n+1} = 1$$

$$\Rightarrow \sum_{i=1}^{n} \frac{i}{n}\pi_1 + \sum_{i=1}^{m} \frac{i}{m}\pi_1 = 1$$

$$\Rightarrow \left(\frac{n(n+1)}{2n} + \frac{m(m+1)}{2m}\right)\pi_1 = 1$$

$$\Rightarrow \frac{n+m+2}{2}\pi_1 = 1$$

$$\Rightarrow \pi_1 = \frac{2}{n+m+2}$$

Now, we compute $\sum_{i=n_0}^{n} \pi_i$:

$$\sum_{i=n_0}^{n} \pi_i$$

$$= \sum_{i=n_0}^{n} \frac{n-i+1}{n}\pi_1$$

$$= \sum_{i=1}^{n-n_0+1} \frac{i}{n}\pi_1$$

$$= \frac{(n-n_0+1)(n-n_0+2)}{2n}\frac{2}{n+m+2}$$

$$= \frac{(n-n_0+1)(n-n_0+2)}{n(n+m+2)}$$

5. [20] An HMM (Hidden Markov Model) with two Hidden States, H_0 and H_1, and two Visible States, V_0 and V_1, is given. The transition probability matrix is denoted by A, the observation probability matrix by B, and the initial probability vector by π.

$$A = \begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix}, B = \begin{bmatrix} 0.8 & 0.2 \\ 0.2 & 0.8 \end{bmatrix}, \pi = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

If the following sequence is generated by this model, update the value of p by performing one step of the EM algorithm. (Assume the initial value of p is $p = \frac{1}{2}$.)

$$0, 1, 0, 1, 1, 0$$

(Hint: Compute the values of α, β, ξ, and γ.)

Solution:
To compute this, we first need to calculate the values of α, β, ξ, and γ.

$$\alpha_1(i) = \pi_i b_i(o_1)$$

$$\alpha_{k+1}(j) = \left[\sum_i \alpha_k(i) a_{ij}\right] b_j(o_{k+1})$$

α	$\alpha_1()$	$\alpha_2()$	$\alpha_3()$	$\alpha_4()$	$\alpha_5()$	$\alpha_6()$
$\alpha(0)$	0.8	$\frac{0.8}{2} \cdot 0.2$	$\frac{0.8}{2^2} \cdot 0.8$	$\frac{0.8}{2^3} \cdot 0.2$	$\frac{0.8}{2^4} \cdot 0.2$	$\frac{0.8}{2^5} \cdot 0.8$
$\alpha(1)$	0	$\frac{0.8}{2} \cdot 0.8$	$\frac{0.8}{2^2} \cdot 0.2$	$\frac{0.8}{2^3} \cdot 0.8$	$\frac{0.8}{2^4} \cdot 0.8$	$\frac{0.8}{2^5} \cdot 0.2$

Table 5: α values

$$\beta_K(i) = 1$$

$$\beta_k(j) = \sum_i \beta_{k+1}(i) a_{ji} b_i(o_{k+1})$$

β	$\beta_1()$	$\beta_2()$	$\beta_3()$	$\beta_4()$	$\beta_5()$	$\beta_6()$
$\beta(0)$	$\frac{1}{2^5}$	$\frac{1}{2^4}$	$\frac{1}{2^3}$	$\frac{1}{2^2}$	$\frac{1}{2}$	1
$\beta(1)$	$\frac{1}{2^5}$	$\frac{1}{2^4}$	$\frac{1}{2^3}$	$\frac{1}{2^2}$	$\frac{1}{2}$	1

Table 6: β values

$$\begin{aligned} \xi_k(i,j) &= \frac{\alpha_k(i) a_{ij} b_j(o_{k+1}) \beta_{k+1}(j)}{\sum_{i'} \sum_{j'} \alpha_k(i') a_{i'j'} b_{j'}(o_{k+1}) \beta_{k+1}(j')} \\ &\propto \alpha_k(i) a_{ij} b_j(o_{k+1}) \beta_{k+1}(j) \\ &\propto \alpha_k(i) b_j(o_{k+1}) \end{aligned}$$

$$\begin{aligned} \gamma_k(i) &= \frac{\alpha_k(i)\beta(i)}{\sum_{i'} \alpha_k(i')\beta(i')} \\ &\propto \alpha_k(i)\beta(i) \end{aligned}$$

ξ	$\xi_1()$	$\xi_2()$	$\xi_3()$	$\xi_4()$	$\xi_5()$	$\xi_6()$
$\xi(0,0)$	0.2	0.04	0.64	0.04	0.04	0.64
$\xi(0,1)$	0.8	0.16	0.16	0.16	0.16	0.16
$\xi(1,0)$	0	0.16	0.16	0.16	0.16	0.16
$\xi(1,1)$	0	0.64	0.04	0.64	0.64	0.04

Table 7: ξ values

γ	$\gamma_1()$	$\gamma_2()$	$\gamma_3()$	$\gamma_4()$	$\gamma_5()$	$\gamma_6()$
$\gamma(0)$	1	0.2	0.8	0.2	0.2	0.8
$\gamma(1)$	0	0.8	0.2	0.8	0.8	0.2

Table 8: γ values

$$\propto \alpha_k(i)$$

$$a_{ij} = \frac{\sum_k \xi_k(i,j)}{\sum_i \gamma_k(i)}$$

Using this relation, p can be written as:

$$p = \frac{\sum_k \xi_k(0,0) + \xi_k(1,1)}{\sum_k \gamma_k(0) + \gamma_k(1)}$$

$$= \frac{0.2 + 0.68 + 0.68 + 0.68 + 0.68 + 0.68}{6} = 0.6$$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
−3.6	.0002	.0002	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001
−3.5	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002
−3.4	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0002
−3.3	.0005	.0005	.0005	.0004	.0004	.0004	.0004	.0004	.0004	.0003
−3.2	.0007	.0007	.0006	.0006	.0006	.0006	.0006	.0005	.0005	.0005
−3.1	.0010	.0009	.0009	.0009	.0008	.0008	.0008	.0008	.0007	.0007
−3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
−2.9	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
−2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
−2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
−2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
−2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
−2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
−2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
−2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
−2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
−2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
−1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
−1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
−1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
−1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
−1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
−1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
−1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
−1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
−1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
−1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
−0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
−0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
−0.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
−0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
−0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
−0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
−0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
−0.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
−0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
−0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641

Figure 3: z-table

df/p	0.40	0.25	0.10	0.05	0.025	0.01	0.005	0.0005
1	0.324920	1.000000	3.077684	6.313752	12.70620	31.82052	63.65674	636.6192
2	0.288675	0.816497	1.885618	2.919986	4.30265	6.96456	9.92484	31.5991
3	0.276671	0.764892	1.637744	2.353363	3.18245	4.54070	5.84091	12.9240
4	0.270722	0.740697	1.533206	2.131847	2.77645	3.74695	4.60409	8.6103
5	0.267181	0.726687	1.475884	2.015048	2.57058	3.36493	4.03214	6.8688
6	0.264835	0.717558	1.439756	1.943180	2.44691	3.14267	3.70743	5.9588
7	0.263167	0.711142	1.414924	1.894579	2.36462	2.99795	3.49948	5.4079
8	0.261921	0.706387	1.396815	1.859548	2.30600	2.89646	3.35539	5.0413
9	0.260955	0.702722	1.383029	1.833113	2.26216	2.82144	3.24984	4.7809
10	0.260185	0.699812	1.372184	1.812461	2.22814	2.76377	3.16927	4.5869
11	0.259556	0.697445	1.363430	1.795885	2.20099	2.71808	3.10581	4.4370
12	0.259033	0.695483	1.356217	1.782288	2.17881	2.68100	3.05454	43178
13	0.258591	0.693829	1.350171	1.770933	2.16037	2.65031	3.01228	4.2208
14	0.258213	0.692417	1.345030	1.761310	2.14479	2.62449	2.97684	4.1405
15	0.257885	0.691197	1.340606	1.753050	2.13145	2.60248	2.94671	4.0728
16	0.257599	0.690132	1.336757	1.745884	2.11991	2.58349	2.92078	4.0150
17	0.257347	0.689195	1.333379	1.739607	2.10982	2.56693	2.89823	3.9651
18	0.257123	0.688364	1.330391	1.734064	2.10092	2.55238	2.87844	3.9216
19	0.256923	0.687621	1.327728	1.729133	2.09302	2.53948	2.86093	3.8834
20	0.256743	0.686954	1.325341	1.724718	2.08596	2.52798	2.84534	3.8495
21	0.256580	0.686352	1.323188	1.720743	2.07961	2.51765	2.83136	3.8193
22	0.256432	0.685805	1.321237	1.717144	2.07387	2.50832	2.81876	3.7921
23	0.256297	0.685306	1.319460	1.713872	2.06866	2.49987	2.80734	3.7676
24	0.256173	0.684850	1.317836	1.710882	2.06390	2.49216	2.79694	3.7454
25	0.256060	0.684430	1.316345	1.708141	2.05954	2.48511	2.78744	3.7251
26	0.255955	0.684043	1.314972	1.705618	2.05553	2.47863	2.77871	3.7066
27	0.255858	0.683685	1.313703	1.703288	2.05183	2.47266	2.77068	3.6896
28	0.255768	0.683353	1.312527	1.701131	2.04841	2.46714	2.76326	3.6739
29	0.255684	0.683044	1.311434	1.699127	2.04523	2.46202	2.75639	3.6594
30	0.255605	0.682756	1.310415	1.697261	2.04227	2.45726	2.75000	3.6460

Figure 4: t-table