In the name of GOD.

Stochastic Process

Sharif University of Technology

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Hamid R. Rabiee

Homework 4 | Estimation Theory | Deadline : 1403/09/17

1. [5] The time interval between the arrival of trains at Sharif Metro Station in the morning follows an exponential distribution with parameter λ. If $x_1, \ldots, x_N$ are independent samples of the time between the arrival of two trains at this station in the morning, Find a *minimal sufficient statistic* for λ.

2. [10] Let $X_1, \ldots, X_N$ be iid with PDF:

$$f(x;\theta) = \begin{cases} \frac{x}{\theta} exp\left(-\frac{x^2}{2\theta}\right), & x > 0 \\ 0, & x \leq 0 \end{cases}$$

   (a) Find a scalar (one-dimensional) *sufficient statistic* for θ using *factorization theorem*.

   (b) Find the *method of moments* estimator of θ.

3. [20] Suppose that the random variables $Y_1, \ldots, Y_n$ satisfy

$$Y_i = \beta x_i + e_i, \quad i = 1, \ldots, n,$$

   where $x_1, \ldots, x_n$ are fixed known constants and $e_1, \ldots, e_n$ are iid samples from $N(0, \sigma^2)$.

   (a) Find the *MLE* of β, and show that it is an unbiased estimator of β.

   (b) Calculate the mean and variance of $S = \frac{\sum Y_i}{\sum x_i}$ as an estimator for β, and then compare it to the *MLE* of β.

4. [10] Let $x_1, \ldots, x_N$ are iid samples from a distribution with PDF as follows:

$$f(x;\theta) = 2e^{-|x-\theta|}$$

   Find the *MLE* of θ.

5. [15] Let $X_1, \ldots, X_N$ be iid with PDF:

$$f(x;\theta) = \begin{cases} e^{(\theta - x)}, & x \geq \theta \\ 0, & x < \theta \end{cases}$$

   (a) Find a *complete sufficient statistic* for θ.

   (b) Use this sufficient statistic and calculate *UMVUE* for θ.

6. [20] Let $x_1, \ldots, x_N$ are iid samples from the following distribution:

$$f(x;\alpha,\beta) = \frac{1}{\beta} e^{-\frac{x-\alpha}{\beta}}, \quad x \geq \alpha\ , \quad \beta > 0$$

Find the $UMVUE$ for α *and* β.

(Hint: Show MLE of these parameters are *complete sufficient statistics*.)

7. [15] Let $x_1, \ldots, x_N$ are iid samples from $U(0, \theta)$, which prior distribution of θ is:

$$\pi(\theta|\alpha,\beta) = \frac{\alpha\beta^{\alpha}}{\theta^{\alpha+1}} \quad \theta \geq \alpha, \beta > 0$$

   (a) Find the MAP estimator for θ.

   (b) Find *Bayes Minimum Loss* estimator for θ. Use *Squared Error Loss*.

8. [5] Suppose $P(\theta;\alpha)$ is a *conjugate prior* for $f(x|\theta)$. Show that the following distribution is a *conjugate prior* for $f(x|\theta)$ too.

$$\sum_{i=1}^{m} \beta_i P(\theta;\alpha_i), \quad s.t. \sum_{i=1}^{m} \beta_i = 1$$