In the name of GOD.

Sharif University of Technology

Stochastic Process

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Homework 3	Point Process, Poisson Process, Gaussian Process	Deadline : 1403/8/21

1. Let X and Z be IID normalized Gaussian random variables. Let $Y = |Z|\text{Sgn}(X)$, where $\text{Sgn}(X)$ is 1 if $X \geq 0$ and -1 otherwise. Show that X and Y are each Gaussian, but are not jointly Gaussian. Sketch the contours of equal joint probability density.

2. A radioactive source emits particles according to a Poisson process of rate 2 particles per minute.

   (a) Compute the probability p_a that the first particle appears some time after 3 minutes and before 5 minutes.

   (b) Compute the probability p_b that exactly one particle is emitted in the time interval from 3 to 5 minutes.

3. Given a normal process $x(t)$ with $\eta_x = 0$ and $R_x(\tau) = 4e^{-2|\tau|}$, we form the random variables $z = x(t+1)$, $w = x(t-1)$:

   (a) Find $E[zw]$ and $E[(z+w)^2]$.

   (b) Find $f_z(z)$, $P\{z < 1\}$, and $f_{zw}(z, w)$.

4. In one of the ancient cities, there was a traditional restaurant that served as the main gathering place for the city's residents. Every morning, all the people lined up in front of the restaurant and entered one by one. Once inside, they chose a table and stayed there until the end of the day. One of the favorite pastimes of the residents of this city is choosing their table at random. Now, suppose Nth person in line wants to enter while $N-1$ people are already seated . This person has two options: can sit at one of the tables that already have people, with the probability of choosing table k (which currently has n_k people seated) given by $\frac{n_k}{N-1+\alpha}$, or alternatively, can choose a new table with a probability of $\frac{\alpha}{N-1+\alpha}$, where n_k is the number of people already seated at table k, and α is a fixed constant. Considering this seating process, if the total population of the city is M, and α is equal to 1, determine the average number of tables occupied in one day. Express the answer in terms of H_n ($H_n = \sum_{i=1}^{n} \frac{1}{i}$).

5. (a) Let $X_1 \sim N(0, \sigma_1^2)$ and $X_2 \sim N(0, \sigma_2^2)$ be independent random variables. Show that $X_1 + X_2$ follows the distribution $N(0, \sigma_1^2 + \sigma_2^2)$.

   (b) Let W_1, W_2 be i.i.d. normalized Gaussian random variables. Show that $a_1W_1 + a_2W_2$ is Gaussian, $\mathcal{N}(0, a_1^2 + a_2^2)$.

   (c) Using the result from part (b), to show that all linear combinations of i.i.d. normalized Gaussian random variables are Gaussian.

6. Earthquakes occur in a given region in accordance with a Poisson process with rate 5 per year.

(a) What is the probability that there will be at least two earthquakes in the first half of 2020?

(b) Assuming that the event in part (a) occurs, what is the probability that there will be no earthquakes during the first 9 months of 2021?

(c) Assuming that the event in part (a) occurs, what is the probability that there will be at least four earthquakes over the first 9 months of the year 2020?

7. A stochastic process $\{X(t), t \geq 0\}$ is said to be stationary if $X(t_1), \ldots, X(t_n)$ has the same joint distribution as $X(t_1 + a), \ldots, X(t_n + a)$ for all $n, a, t_1, \ldots, t_n$.

   (a) Prove that a necessary and sufficient condition for a Gaussian process to be stationary is that $\mathrm{Cov}(X(s), X(t))$ depends only on $t - s$, $s \leq t$, and $\mathbb{E}[X(t)] = c$.

   (b) Let $\{X(t), t \geq 0\}$ be Brownian motion and define

$$V(t) = e^{-\alpha t/2} X(a e^{\alpha t}).$$

   Show that $\{V(t), t \geq 0\}$ is a stationary Gaussian process. It is called the Ornstein-Uhlenbeck process. (**Hint:** Brownian motion is a Gaussian process W_t such that $\mathbb{E}[W_t] = 0$ and $\mathrm{cov}(W_t, W_s) = \min(t, s)$.)

8. Let X_t and Y_t represent two independent Poisson processes with arrival rates λ_1 and λ_2, respectively, where these rates indicate the hourly arrival rate of customers at stores 1 and 2.

   (a) What is the probability that a customer arrives at store 1 before any customers arrive at store 2?

   (b) What is the probability that, during the first hour, the combined total number of customers arriving at both stores is exactly four?

   (c) Given that exactly four customers arrived across the two stores, what is the probability that all four arrived exclusively at store 1?

   (d) Let T denote the arrival time of the first customer at store 2. Then, X_T represents the count of customers at store 1 by the time the first customer arrives at store 2. Determine the probability distribution of X_T.

9. Suppose $X(t)$ is a Gaussian process, with $X(0) = 0$ with probability 1. Suppose that $X_t + X_s \sim \mathcal{N}\left(0, \sqrt{|t - s|}\right)$.

   (a) Calculate the auto-covariance function.

   (b) Calculate the distribution of $(X(t_1), X(t_2), \ldots, X(t_n))$.

   (c) Prove that such a process doesn't exist.