# 1)

## a)

$$\begin{aligned}
R_{yy}(t_1, t_2) &= E[Y(t_1)Y(t_2)] \\
&= E[X(t_1 + t_2)X(t_2 + t_1)] - \alpha^2 E[X(t_1)X(t_2)] \\
&\quad - \alpha E[X(t_1 + t_2)X(t_2)] - \alpha E[X(t_2 + t_1)X(t_1)] \\
&= R_{xx}(t_2 - t_1) - \alpha^2 R_{xx}(t_2 - t_1) - \alpha R_{xx}(t_2 - t_1 - t_2) \\
&= (1 - \alpha^2)R_{xx}(\tau) - \alpha R_{xx}(t_2 - t_1 + t_2)
\end{aligned}$$

$$\begin{aligned}
E[Y(t)] &= E[X(t + \tau)] - \alpha X(t) \\
&= \mu_x - \alpha\mu_x = (1 - \alpha)\mu_x
\end{aligned}$$

$M_x$ is independent of $\tau$ because $X(t)$ is WSS. $Y(t)$ is WSS because $R_{xx}$ depends on $\tau$ and $\mu_x$ is constant.

## b)

$$\begin{aligned}
C_{xy}(t_1, t_2) &= E[X(t_1)Y(t_2)] - E[X(t_1)]E[Y(t_2)] \\
&= E[X(t_1)X(t_2 + \tau)] - \alpha E[X(t_1)]E[X(t_2)] \\
&= (1 - \alpha)\mu_x^2
\end{aligned}$$

$$C_{xy} = R_{xx}(\tau + t_1) - \alpha R_{xx}(\tau) - (1 - \alpha)\mu_x^2$$

$C_{xy}$ depends on $\tau$ (with $t_1$ fixed). $X, Y$ are jointly WSS.

## 2)

### a) Discrete Time Process

$$E[X_n] = p \cdot 1 + (1-p) \cdot 0 = p$$

$$E[Y_n] = \frac{1}{2}E[X_n] + \frac{1}{2}E[X_{n-1}] = \frac{p}{2} + \frac{p}{2} = p \quad \text{(since } X_n \text{ is iid)}$$

$$Var[Y_n] = E[Y_n^2] - (E[Y_n])^2$$

$$E[Y_n^2] = \frac{1}{4}E[X_n^2] + \frac{1}{2}E[X_n X_{n-1}] + \frac{1}{4}E[X_{n-1}^2]$$

$$E[X_n^2] = p \cdot 1 + 0 \cdot (1-p) = p$$

$$E[X_n^2] = \frac{1}{4}p + \frac{1}{2}p^2 + \frac{1}{4}p = \frac{1}{2}p(1+p)$$

$$Var[Y_n] = \frac{p^2}{2} + \frac{p^2}{2} - p^2 = \frac{1}{2}p - \frac{1}{2}p^2 = \frac{1}{2}p(1-p)$$

### b) Autocorrelation

$R_{yy}(k) = E[Y_n Y_{n+k}] = E[Y_n]E[Y_{n+k}] \quad \text{if } k > 1 = p^2 \quad \text{for } k > 1 \text{ (because iid)}$

Let us also calculate $R_{yy}(k)$ for $k = 1$:

$$R_{yy}(1) = E[Y_n Y_{n+1}] = \frac{1}{4}E[X_n X_{n+1}] + E[Y_n]E[Y_{n+1}] = \frac{1}{4}(p^2 + p^2 + p^2) = \frac{3}{4}p(1+3p)$$

$$R_{yy}(0) = E[Y_n^2] = \frac{p^2}{2} + \frac{p}{2}$$

$$R_{yy}(k) = \begin{cases} \frac{p^2}{2} + \frac{p}{2} & \text{for } k = 0 \\ \frac{3}{4}p + \frac{p^2}{4} & \text{for } k = 1 \\ p^2 & \text{for } k > 1 \end{cases}$$

Solution:

Consider a WSS (Wide-Sense Stationary) random sequence $X[n]$ with: - Mean function $\mu_X$ (a constant) - Correlation function $R_{XX}[m]$

The random process $X(t)$ is defined as:

$$X(t) = \sum_{n=-\infty}^{\infty} X[n] \frac{\sin \pi(t - nT)/T}{\pi(t - nT)/T}$$

Part (a) Find $\mu_X(t)$ in terms of $\mu_X$.

The mean of the process $X(t)$, denoted as $\mu_X(t) = E[X(t)]$, is given by:

$$\mu_X(t) = E\left[\sum_{n=-\infty}^{\infty} X[n] \frac{\sin \pi(t - nT)/T}{\pi(t - nT)/T}\right]$$

Since expectation is a linear operator:

$$\mu_X(t) = \sum_{n=-\infty}^{\infty} E[X[n]] \frac{\sin \pi(t - nT)/T}{\pi(t - nT)/T}$$

Given $E[X[n]] = \mu_X$ for all $n$ (because $X[n]$ is WSS):

$$\mu_X(t) = \mu_X \sum_{n=-\infty}^{\infty} \frac{\sin \pi(t - nT)/T}{\pi(t - nT)/T}$$

The sum:

$$\sum_{n=-\infty}^{\infty} \frac{\sin \pi(t - nT)/T}{\pi(t - nT)/T} = 1$$

This is due to the property of the Dirichlet kernel (sinc function as a sampling reconstruction kernel).

Thus:

$$\mu_X(t) = \mu_X$$

Part (b) Find $R_{XX}(t_1, t_2)$ in terms of $R_{XX}[m]$. Is $X(t)$ WSS?

The autocorrelation function $R_{XX}(t_1, t_2)$ is defined as:

$$R_{XX}(t_1, t_2) = E[X(t_1)X(t_2)]$$

Substituting the definition of $X(t)$:

$$R_{XX}(t_1, t_2) = E\left[\left(\sum_{n=-\infty}^{\infty} X[n] \frac{\sin \pi(t_1 - nT)/T}{\pi(t_1 - nT)/T}\right)\left(\sum_{m=-\infty}^{\infty} X[m] \frac{\sin \pi(t_2 - mT)/T}{\pi(t_2 - mT)/T}\right)\right]$$

Expanding the expectation:

$$R_{XX}(t_1, t_2) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{\sin \pi(t_1 - nT)/T}{\pi(t_1 - nT)/T} \frac{\sin \pi(t_2 - mT)/T}{\pi(t_2 - mT)/T} E[X[n]X[m]]$$

Since $X[n]$ is WSS, $E[X[n]X[m]] = R_{XX}[n-m]$:

$$R_{XX}(t_1,t_2) = \sum_{k=-\infty}^{\infty} R_{XX}[k] \sum_{n=-\infty}^{\infty} \frac{\sin \pi(t_1 - nT)/T}{\pi(t_1 - nT)/T} \frac{\sin \pi(t_2 - (n-k)T)/T}{\pi(t_2 - (n-k)T)/T}$$

Using the orthogonality of the sinc functions:

$$R_{XX}(t_1,t_2) = \sum_{k=-\infty}^{\infty} R_{XX}[k] \frac{\sin \pi(t_1 - t_2 - kT)/T}{\pi(t_1 - t_2 - kT)/T}$$

Now, for $X(t)$ to be WSS, $R_{XX}(t_1,t_2)$ should depend only on $\tau = t_1 - t_2$:

$$R_{XX}(t_1,t_2) = R_{XX}(\tau)$$

Thus, $X(t)$ is WSS if $R_{XX}(t_1,t_2)$ only depends on $\tau$, which is satisfied here since it depends on $\tau$ through $t_1 - t_2$.

Conclusion 1. $\mu_X(t) = \mu_X$ 2. $R_{XX}(t_1,t_2)$ is derived in terms of $R_{XX}[m]$, and $X(t)$ is WSS.

Let $N(t)$ equal the number of zero crossings in the interval $(0,t)$ with $t \geq 0$.

$$P(N = n) = \frac{(\lambda t)^n e^{-\lambda t}}{n!} \tag{1}$$

$$\begin{aligned} P[X(t) = 1] &= P[N = \text{even number}] \\ &= e^{-\lambda t}\left[1 + \frac{(\lambda t)^2}{2!} + \frac{(\lambda t)^4}{4!} + \cdots\right] \\ &= e^{-\lambda t}\cosh \lambda t \end{aligned}$$

$$\begin{aligned} P[X(t) = -1] &= P[N = \text{odd number}] \\ &= e^{-\lambda t}\left[\lambda t + \frac{(\lambda t)^3}{3!} + \frac{(\lambda t)^5}{5!} + \cdots\right] \\ &= e^{-\lambda t}\sinh \lambda t \end{aligned}$$

The expected value is

$$E[X(t)|X(0) = 1] = e^{-\lambda t}\cosh \lambda t - e^{-\lambda t}\sinh \lambda t = e^{-2\lambda t}$$

Note that the expected value decays toward $x = 0$ for large $t$. That happens because the influence of knowing the value at $t = 0$ decays exponentially.

The autocorrelation function is computed by finding $R(t_1, t_2) = E[X(t_1)X(t_2)]$. Let $x_0 = -1$ and $x_1 = 1$ denote the two values that $X$ can attain. For the moment assume that $t_2 \geq t_1$. Then

$$R(t_1, t_2) = \sum_{j=0}^{1}\sum_{k=0}^{1} x_j x_k P[X(t_1) = x_k]P[X(t_2) = x_j|X(t_1) = x_k]$$

The first term in each product is given above. To find the conditional probabilities we take note of the fact that the number of sign changes in $t_2 - t_1$ is a Poisson process. Hence, in a manner that is similar to the analysis above,

$$P[X(t_2) = 1|X(t_1) = 1] = P[X(t_2) = -1|X(t_1) = -1] = e^{-\lambda(t_2 - t_1)}\cosh \lambda(t_2 - t_1)$$

$$P[X(t_2) = -1|X(t_1) = 1] = P[X(t_2) = 1|X(t_1) = -1] = e^{-\lambda(t_2 - t_1)}\sinh \lambda(t_2 - t_1)$$

Hence

$$R(t_1, t_2) = e^{-\lambda t_1}\cosh \lambda t_1\left[e^{-\lambda(t_2 - t_1)}\cosh \lambda(t_2 - t_1) - e^{-\lambda(t_2 - t_1)}\sinh \lambda(t_2 - t_1)\right]$$

$$-e^{-\lambda t_1} \sinh \lambda t_1 \left[ e^{-\lambda(t_2-t_1)} \cosh \lambda(t_2 - t_1) - e^{-\lambda(t_2-t_1)} \sinh \lambda(t_2 - t_1) \right]$$

After some algebra this reduces to

$$R(t_1, t_2) = e^{-\lambda(t_2-t_1)} \quad \text{for } t_2 \geq t_1$$

A parallel analysis applies to the case $t_2 \leq t_1$, so that

$$R(t_1, t_2) = e^{-\lambda|t_2-t_1|}$$

The autocorrelation for the telegraph signal depends only upon the *time difference*, not the location of the time interval. We will see soon that this is a very important characteristic of stationary random processes.

We can now remove condition (3) on the telegraph process. Let $Y(t) = AX(t)$ where $A$ is a random variable independent of $X$ that takes on the values $\pm 1$ with equal probability. Then $Y(0)$ will equal $\pm 1$ with equal probability, and the telegraph process will no longer have the restriction of being positive at $t = 0$.

Since $A$ and $X$ are independent, the autocorrelation for $Y(t)$ is given by

$$E[Y(t_1)Y(t_2)] = E[A^2]E[X(t_1)X(t_2)] = e^{-\lambda|t_2-t_1|}$$

since $E[A^2] = 1$.

To prove: $X(t)$ is correlation ergodic.
i.e., To prove:

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} X(t)X(t+\tau)\,dt = R(\tau)$$

$$\begin{aligned}
\text{R.H.S} &= R(\tau)\\
&= E[X(t+\tau)X(t)]\\
&= E[10\cos(100t+100\tau+\theta)\cdot 10\cos(100t+\theta)]\\
&= 100E[\cos(100t+100\tau+\theta)\cos(100t+\theta)]\\
&= \frac{100}{2}E[\cos(200t+100\tau+2\theta)+\cos(100\tau)]\\
&= 50E[\cos(200t+100\tau+2\theta)]+50E[\cos(100\tau)]\\
&= 50E[\cos(200t+100\tau+2\theta)]+50\cos(100\tau) \qquad \cdots(1)
\end{aligned}$$

$$E[\cos(200t+100\tau+2\theta)] = \int_{-\pi}^{\pi}\cos(200t+100\tau+2\theta)\frac{1}{2\pi}\,d\theta = 0$$

$$\left[\int_{-\pi}^{\pi}\cos(\omega t+n\theta)\,d\theta = 0,\ \text{where } n \text{ is an integer, } n\neq 0\right]$$

$$(1) \implies R(\tau) = 50\cdot 0 + 50\cos(100\tau) = 50\cos(100\tau)$$

$$\text{L.H.S} = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} X(t)X(t+\tau)\,dt$$

$$\begin{aligned}
&= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} 10\cos(100t+100\tau+\theta)\cdot 10\cos(100t+\theta)\,dt\\
&= \lim_{T\to\infty}\frac{100}{2T}\int_{-T}^{T} \cos(100t+100\tau+\theta)\cos(100t+\theta)\,dt\\
&= \lim_{T\to\infty}\frac{50}{2T}\int_{-T}^{T} [\cos(200t+100\tau+2\theta)+\cos(100\tau)]\,dt
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{T\to\infty} \frac{25}{T} \int_{-T}^{T} \cos(200t + 100\tau + 2\theta)\, dt + \lim_{T\to\infty} \frac{25}{T} \int_{-T}^{T} \cos(100\tau)\, dt \\
&= \lim_{T\to\infty} \frac{25}{T} \left[ \frac{\sin(200T + 100\tau + 2\theta)}{200} \right]_{-T}^{T} + \lim_{T\to\infty} \frac{25}{T} \cos(100\tau)\, [t]_{-T}^{T} \\
&= \lim_{T\to\infty} \frac{1}{8T} \left[ \sin(200T + 100\tau + 2\theta) - \sin(-200T + 100\tau + 2\theta) \right] + \lim_{T\to\infty} \frac{25}{T} \cos(100\tau)(2T) \\
&= \lim_{T\to\infty} \frac{1}{8T} \left[ 2\cos\left( \frac{2(100\tau + 2\theta)}{2} \right) \sin\left( \frac{400T}{2} \right) \right] + 50\cos(100\tau) \\
&= \lim_{T\to\infty} \frac{1}{4T} \cos(100\tau + 2\theta) \sin(200T) + 50\cos(100\tau) \\
&= \frac{\cos(100\tau + 2\theta)}{4} \lim_{T\to\infty} \frac{\sin(200T)}{T} + 50\cos(100\tau) \\
&= \frac{\cos(100\tau + 2\theta)}{4} (0) + 50\cos(100\tau) \\
&= 0 + 50\cos(100\tau) = 50\cos(100\tau)
\end{aligned}
$$

$$\text{L.H.S} = \text{R.H.S}$$

Hence, $X(t)$ is correlation ergodic.