In the name of GOD.

Sharif University of Technology

Stochastic Process

Fall 2024
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Solution Homework 1 | Review of Probability | Deadline : 1403/07/20

1. (a) The probability that the dart falls within the σ-radius circle centered at $(0,0)$ can be calculated as follows:

$$\Pr(x^2 + y^2 \leq \sigma^2) = \int_0^{2\pi} \int_0^{\sigma} \frac{r}{2\sigma^2} e^{-\frac{r^2}{2\sigma^2}} \, dr \, d\theta = 1 - e^{-1}.$$

(b) The probability that the dart falls in the first quadrant $(x > 0, y > 0)$, due to the independence of x and y and the symmetry of Gaussian distribution in each quadrant, is:

$$\Pr(x > 0, y > 0) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}.$$

(c) The conditional probability that the dart falls within the σ-radius circle centered at $(0,0)$ given that it hits in the first quadrant is:

$$\Pr(x^2 + y^2 \leq \sigma^2 \mid x > 0, y > 0) = \frac{\Pr(x^2 + y^2 \leq \sigma^2 \cap x > 0 \cap y > 0)}{\Pr(x > 0, y > 0)}.$$

Since $x^2 + y^2 \leq \sigma^2$ is symmetric across all four quadrants, we get:

$$\Pr(x^2 + y^2 \leq \sigma^2 \mid x > 0, y > 0) = \frac{1 - e^{-1}}{4} \div \frac{1}{4} = 1 - e^{-1}.$$

(d) Let $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ be the polar coordinates. We have:

$$\Pr[0 \leq r \leq R, 0 \leq \theta \leq \Theta] = \int_0^{\Theta} \int_0^{R} f_{R,\Theta}(r,\theta) \, r \, dr \, d\theta.$$

The joint probability density function $P_{R,\Theta}(r,\theta)$ can be derived as:

$$P_{R,\Theta}(r,\theta) = \frac{r}{2\sigma^2} e^{-\frac{r^2}{2\sigma^2}}, \quad 0 \leq r \leq \infty, \, 0 \leq \theta \leq 2\pi.$$

2. **Proof:**

We want to prove the following equation for the conditional expectation:

$$E\left[Y|X \leq 0\right] = \frac{\int_{-\infty}^{0} E\left[Y|X = x\right] f_X(x) \, dx}{F_X(0)}. \quad (1)$$

Recall that the conditional expectation of Y given $X \leq 0$ is defined as:

$$E[Y|X \leq 0] = \int_{-\infty}^{\infty} y\, f_{Y|X\leq 0}(y)\, dy.$$

Next, we can express the conditional density function $f_{Y|X\leq 0}(y)$ in terms of the joint density $f_{X,Y}(x,y)$ and the marginal density $f_X(x)$:

$$f_{Y|X\leq 0}(y) = \frac{\int_{-\infty}^{0} f_{X,Y}(x,y)\, dx}{\Pr(X \leq 0)}.$$

Substitute the definition of $f_{X,Y}(x,y) = f_{Y|X}(y|x) f_X(x)$:

$$f_{Y|X\leq 0}(y) = \frac{\int_{-\infty}^{0} f_{Y|X}(y|x) f_X(x)\, dx}{F_X(0)}.$$

Now, we can calculate the conditional expectation $E[Y|X \leq 0]$:

$$E[Y|X \leq 0] = \int_{-\infty}^{\infty} y \left[\frac{\int_{-\infty}^{0} f_{Y|X}(y|x) f_X(x)\, dx}{F_X(0)} \right] dy.$$

Using the property of conditional expectation $E[Y|X = x] = \int_{-\infty}^{\infty} y f_{Y|X}(y|x)\, dy$, we can rewrite the inner integral as:

$$E[Y|X \leq 0] = \frac{\int_{-\infty}^{0} E[Y|X = x] f_X(x)\, dx}{F_X(0)}.$$

This completes the proof of the given equation:

$$E\left[Y|X \leq 0\right] = \frac{\int_{-\infty}^{0} E\left[Y|X = x\right] f_X(x)\, dx}{F_X(0)}.$$

3. **Solution:**

   Let X be a random variable with cumulative distribution function (CDF) $F_X(x)$ and probability density function (PDF) $f_X(x)$.

   We want to find the conditional CDF and PDF of X given that $a < X \leq b$.

   1. Conditional CDF:

   The conditional CDF of X given $a < X \leq b$, denoted as $F_{X|a<X\leq b}(x)$, is defined as:

$$F_{X|a<X\leq b}(x) = \Pr(X \leq x \mid a < X \leq b), \quad a < x \leq b.$$

   By definition of conditional probability, we have:

$$F_{X|a<X\leq b}(x) = \frac{\Pr(a < X \leq x)}{\Pr(a < X \leq b)}.$$

Since $\Pr(a < X \leq x) = F_X(x) - F_X(a)$ and $\Pr(a < X \leq b) = F_X(b) - F_X(a)$, we can rewrite the conditional CDF as:

$$F_{X|a<X\leq b}(x) = \frac{F_X(x) - F_X(a)}{F_X(b) - F_X(a)}, \quad a < x \leq b.$$

2. Conditional PDF:

The conditional PDF of X given $a < X \leq b$, denoted as $f_{X|a<X\leq b}(x)$, is obtained by differentiating the conditional CDF $F_{X|a<X\leq b}(x)$ with respect to x:

$$f_{X|a<X\leq b}(x) = \frac{d}{dx} F_{X|a<X\leq b}(x), \quad a < x \leq b.$$

Taking the derivative of $F_{X|a<X\leq b}(x)$:

$$f_{X|a<X\leq b}(x) = \frac{f_X(x)}{F_X(b) - F_X(a)}, \quad a < x \leq b.$$

This is the desired conditional PDF of X given $a < X \leq b$.

Final Results:

- **Conditional CDF**:

$$F_{X|a<X\leq b}(x) = \frac{F_X(x) - F_X(a)}{F_X(b) - F_X(a)}, \quad a < x \leq b.$$

- **Conditional PDF**:

$$f_{X|a<X\leq b}(x) = \frac{f_X(x)}{F_X(b) - F_X(a)}, \quad a < x \leq b.$$

4. The random variables x and y are independent with exponential densities:

$$f_X(x) = \alpha e^{-\alpha x} U(x), \quad f_Y(y) = \beta e^{-\beta y} U(y),$$

where $U(x)$ is the unit step function.

Find the densities of the following random variables:

(a) **Density of $2x + y$:**

Let $Z = 2x + y$. Since x and y are independent random variables, we can find the PDF of Z using the convolution formula:

$$f_Z(z) = \int_0^{\infty} f_X(x) f_Y(z - 2x)\, dx.$$

Substituting the PDFs of x and y, we get:

$$f_Z(z) = \int_0^{\frac{z}{2}} \alpha e^{-\alpha x} \beta e^{-\beta(z-2x)}\, dx.$$

Simplifying the integral, we obtain:

$$f_Z(z) = \alpha\beta e^{-\beta z} \int_0^{\frac{z}{2}} e^{(2\beta-\alpha)x}\, dx, \quad z \geq 0.$$

Evaluating the integral:

$$f_Z(z) = \alpha\beta e^{-\beta z} \left[\frac{e^{(2\beta-\alpha)x}}{2\beta - \alpha} \right]_0^{\frac{z}{2}}.$$

$$f_Z(z) = \frac{\alpha\beta}{2\beta - \alpha} e^{-\beta z} \left(e^{(2\beta-\alpha)\frac{z}{2}} - 1 \right), \quad z \geq 0.$$

(b) **Density of $x - y$:**
Let $W = x - y$. Since x and y are independent, we can find the PDF of W using the properties of independent random variables and their characteristic functions or moment generating functions. Alternatively, the convolution method can be applied again, resulting in a Laplace distribution:

$$f_W(w) = \begin{cases} \alpha\beta e^{-\beta w} \int_0^\infty e^{-(\alpha+\beta)x} dx, & w \geq 0, \\ \alpha\beta e^{\alpha w} \int_0^\infty e^{-(\alpha+\beta)y} dy, & w < 0. \end{cases}$$

(c) **Density of $\frac{y}{x}$:**
Let $U = \frac{y}{x}$. We find the joint density of x and y, and then use the transformation of variables technique. For $u = \frac{y}{x}$, the joint PDF can be expressed as:

$$f_U(u) = \int_0^\infty \alpha e^{-\alpha x} \beta e^{-\beta u x} x \, dx, \quad u \geq 0.$$

Simplifying, we obtain:

$$f_U(u) = \frac{\alpha\beta}{(\alpha + \beta u)^2}, \quad u \geq 0.$$

(d) **Density of $\max(x, y)$:**
Let $M = \max(x, y)$. We find the CDF of M:

$$F_M(m) = \Pr(\max(x, y) \leq m) = \Pr(x \leq m, y \leq m).$$

Using the independence of x and y:

$$F_M(m) = F_X(m) F_Y(m) = (1 - e^{-\alpha m})(1 - e^{-\beta m}), \quad m \geq 0.$$

Differentiating $F_M(m)$ with respect to m, we get the PDF:

$$f_M(m) = (\alpha + \beta) e^{-(\alpha+\beta)m} - \alpha\beta m e^{-(\alpha+\beta)m}, \quad m \geq 0.$$

(e) **Density of $\min(x, y)$:**
Let $N = \min(x, y)$. We find the CDF of N:

$$F_N(n) = \Pr(\min(x, y) \leq n) = 1 - \Pr(x > n, y > n).$$

Using the independence of x and y:

$$F_N(n) = 1 - \Pr(x > n) \Pr(y > n) = 1 - e^{-\alpha n} e^{-\beta n}, \quad n \geq 0.$$

Differentiating $F_N(n)$ with respect to n, we get the PDF:

$$f_N(n) = (\alpha + \beta) e^{-(\alpha+\beta)n}, \quad n \geq 0.$$

5. **Solution:**

We want to estimate $Y' = a_1X_1 + a_2X_2$ such that the mean squared error (MSE) between Y and Y' is minimized. That is, we want to find a_1 and a_2 such that:

$$p = E\left[(Y - Y')^2\right] = E\left[(Y - a_1X_1 - a_2X_2)^2\right]$$

is minimized.

Step 1: Expanding the Mean Squared Error We start by expanding the squared error:

$$p = E\left[Y^2 - 2Y(a_1X_1 + a_2X_2) + (a_1X_1 + a_2X_2)^2\right].$$

Using the linearity of expectation, we can separate the expectation as:

$$p = E[Y^2] - 2a_1E[YX_1] - 2a_2E[YX_2] + a_1^2E[X_1^2] + 2a_1a_2E[X_1X_2] + a_2^2E[X_2^2].$$

Step 2: Finding the Minimum To find the values of a_1 and a_2 that minimize p, we take the partial derivatives of p with respect to a_1 and a_2 and set them equal to zero:

$$\frac{\partial p}{\partial a_1} = -2E[YX_1] + 2a_1E[X_1^2] + 2a_2E[X_1X_2] = 0,$$
$$\frac{\partial p}{\partial a_2} = -2E[YX_2] + 2a_1E[X_1X_2] + 2a_2E[X_2^2] = 0.$$

Simplifying these equations, we get a system of linear equations:

$$\begin{cases} a_1E[X_1^2] + a_2E[X_1X_2] = E[YX_1], \\ a_1E[X_1X_2] + a_2E[X_2^2] = E[YX_2]. \end{cases}$$

Step 3: Solving the Linear System We can express the system of equations in matrix form as:

$$\begin{bmatrix} E[X_1^2] & E[X_1X_2] \\ E[X_1X_2] & E[X_2^2] \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} E[YX_1] \\ E[YX_2] \end{bmatrix}.$$

Let:

$$\mathbf{A} = \begin{bmatrix} E[X_1^2] & E[X_1X_2] \\ E[X_1X_2] & E[X_2^2] \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} E[YX_1] \\ E[YX_2] \end{bmatrix}.$$

The solution for $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ is given by:

$$\mathbf{a} = \mathbf{A}^{-1}\mathbf{b}.$$

Step 4: Minimum Error The minimum mean squared error P_{min} can be computed as:

$$P_{min} = E[Y^2] - \mathbf{b}^T\mathbf{A}^{-1}\mathbf{b}.$$

Final Solution 1. The values of a_1 and a_2 are given by:

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \mathbf{A}^{-1}\mathbf{b}.$$

2. The minimum error is:

$$P_{min} = E[Y^2] - \mathbf{b}^T\mathbf{A}^{-1}\mathbf{b}.$$

This is the desired solution in terms of the moments of X_1, X_2, and Y.

6. **Solution:**

We have a coin with a probability of 0.1 for heads and 0.9 for tails. We toss this coin 100 times. Let X be the number of heads obtained in 100 trials. Then X follows a Binomial distribution with parameters $n = 100$ and $p = 0.1$, i.e.,

$$X \sim \text{Binomial}(100, 0.1).$$

The mean and variance of X can be calculated as follows:

$$\mu = E[X] = n \cdot p = 100 \cdot 0.1 = 10,$$
$$\sigma^2 = \text{Var}(X) = n \cdot p \cdot (1 - p) = 100 \cdot 0.1 \cdot 0.9 = 9.$$

Now, let's solve each part using the given inequalities:

Part A: Using Markov's Inequality

Markov's inequality states that for a non-negative random variable X and $a > 0$:

$$\Pr(X \geq a) \leq \frac{E[X]}{a}.$$

We want to find the upper bound for the probability that the number of heads is at least 20, i.e., $\Pr(X \geq 20)$.

Applying Markov's inequality:

$$\Pr(X \geq 20) \leq \frac{E[X]}{20}.$$

Substitute $E[X] = 10$:

$$\Pr(X \geq 20) \leq \frac{10}{20} = 0.5.$$

So, the probability that the number of heads is at least 20 is at most 0.5.

Part B: Using Chebyshev's Inequality

Chebyshev's inequality states that for any random variable X with mean μ and variance σ^2, and for any $k > 0$:

$$\Pr(|X - \mu| \geq k\sigma) \leq \frac{1}{k^2}.$$

We want to find the probability that the number of heads is at least 20. First, calculate k such that $|X - \mu| \geq 20 - 10 = 10$.

Using $\sigma = \sqrt{9} = 3$:

$$k = \frac{10}{3}.$$

Applying Chebyshev's inequality:

$$\Pr(|X - 10| \geq 10) \leq \frac{1}{\left(\frac{10}{3}\right)^2} = \frac{1}{\frac{100}{9}} = \frac{9}{100} = 0.09.$$

So, the probability that the number of heads is at least 20 is at most 0.09.

7. **Solution:**

Let $X_1, X_2, \ldots, X_n$ be a sequence of independent and identically distributed (iid) random variables with:

$$E[X_i] = 0 \quad \text{and} \quad \text{Var}(X_i) = \sigma^2.$$

We define the following quantities:

$$S_n = X_1 + X_2 + \cdots + X_n,$$
$$Y_n = \frac{S_n}{\sigma\sqrt{n}} - \frac{S_{2n}}{\sigma\sqrt{2n}}.$$

We want to find the limit of the sequence Y_n as $n \to \infty$ using the Central Limit Theorem (CLT).

Step 1: Apply the Central Limit Theorem According to the Central Limit Theorem, as $n \to \infty$, the normalized sum $\frac{S_n}{\sigma\sqrt{n}}$ converges in distribution to a standard normal random variable $Z \sim \mathcal{N}(0,1)$:

$$\frac{S_n}{\sigma\sqrt{n}} \xrightarrow{\text{d}} Z_1, \quad \text{as } n \to \infty,$$

$$\frac{S_{2n}}{\sigma\sqrt{2n}} \xrightarrow{\text{d}} Z_2, \quad \text{as } n \to \infty,$$

where Z_1 and Z_2 are independent standard normal random variables.

Step 2: Limit of Y_n Now, consider the sequence Y_n:

$$Y_n = \frac{S_n}{\sigma\sqrt{n}} - \frac{S_{2n}}{\sigma\sqrt{2n}}.$$

As $n \to \infty$, using the results from Step 1, we have:

$$Y_n \xrightarrow{\text{d}} Z_1 - Z_2.$$

Since Z_1 and Z_2 are independent standard normal random variables, the difference $Z_1 - Z_2$ follows a normal distribution with mean 0 and variance:

$$\mathrm{Var}(Z_1 - Z_2) = \mathrm{Var}(Z_1) + \mathrm{Var}(Z_2) = 1 + 1 = 2.$$

Therefore, as $n \to \infty$, the limiting distribution of Y_n is:

$$Y_n \xrightarrow{\mathrm{d}} \mathcal{N}(0, 2).$$

Final Result The limit of the sequence Y_n as $n \to \infty$ is a normal distribution with mean 0 and variance 2:

$$Y_n \xrightarrow{\mathrm{d}} \mathcal{N}(0, 2).$$