In the name of GOD.

Sharif University of Technology

Stochastic Process

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Homework 1 | Review of Probability | Deadline : 1403/07/20

1. A dart is thrown at random at a wall. Let (x, y) denote the Cartesian coordinates of the point in the wall pierced by the dart. Suppose that x and y are statistically independent Gaussian random variables, each with mean zero and variance $2\sigma^2$, i.e.

$$f_X(\alpha) = f_Y(\alpha) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\alpha^2}{2\sigma^2}}. \tag{1}$$

   (a) Find the probability that the dart will fall within the σ-radius circle centered at the point $(0, 0)$.

   (b) Find the probability that the dart will hit in the first quadrant $(x > 0, y > 0)$.

   (c) Find the conditional probability that the dart will fall within the σ-radius circle centered at $(0, 0)$ given that the dart hits in the first quadrant.

   (d) Let $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ be the polar coordinates associated with (x, y). Find $\Pr[0 \le r \le R, 0 \le \theta \le \Theta]$ and obtain $P_{R,\Theta}(r, \theta)$.

   This observation leads to a widely used algorithm for generating Gaussian random variables.

2. Prove that :

$$E[Y|X \le 0] = \frac{\int_{-\infty}^{0} E[Y|X = x] . f_x(x)dx}{F_x(0)} \tag{2}$$

3. Find conditional CDF and PDF of random variable X condition to $a < X \le b$ in terms of $F_x(x)$ and $f_x(x)$

4. The random variables x and y are independent with exponential densities

$$f_X(x) = \alpha e^{-\alpha x} U(x), \quad f_Y(y) = \beta e^{-\beta y} U(y)$$

   where $U(x)$ is the unit step function.

   Find the densities of the following random variables:

   (a) $2x + y$

   (b) $x - y$

   (c) $\frac{y}{x}$

   (d) $\max(x, y)$

   (e) $\min(x, y)$

5. The three random variables X_1, X_2, Y are assumed, and we want to estimate $Y' = a_1X_1 + a_2X_2$ by viewing the values of X_1 and X_2 so that $p = E[(Y - Y')^2]$ is minimized. Find the values of a_1 and a_2 as well as the minimum error P_{min} in terms of moments of X_1, X_2, Y.

6. We have a coin with a probability of 0.1 head and a probability of 0.9 tail if we throw this coin 100 times:

   A) Using Markov's inequality, show that the probability of a head falling at least 20 times is maximum 0.5.

   B) Using Chebyshev's inequality, show that the probability of a head falling at least 20 times is 0.09.

7. Suppose $X_1, X_2, ..., X_n$ are sequences of iid random variables with mean equal to zero and variance $= \sigma^2$. we define $Y_n = \frac{s_n}{\sigma * \sqrt{n}} - \frac{s_{2n}}{\sigma * \sqrt{2n}}, S_n = X_1 + X_2 + ... + X_n$ Using the central limit theorem, obtain the limit of the sequence Y_n when $n \to \infty$.