Exercise 3.

(a) (Rossi 4.2.3) Let $X_1, \ldots, X_n$ be a sample of iid Gamma$(4, \theta)$ random variables with $\theta \in \Theta = (0, \infty)$.

(i) Determine the likelihood function $L(\theta)$.

(ii) Use the Neyman-Fisher factorization theorem to determine a sufficient statistic S for θ.

(b) (Rossi 4.3.6) Let $X_1, \ldots, X_n$ be a sample of iid Gamma$(4, \theta)$ random variables with $\theta \in \Theta = (0, \infty)$.

(i) Determine a complete sufficient statistic for θ

(ii) Determine Fisher's Information number for θ

(iii) Determine the Cramér-Rao lower bound for unbiased estimators of θ.

(iv) Determine an UMVUE for 4θ

Solution:

(a) (Rossi 4.2.3 solution)

(i) We have that the pdf of X_j can be written as

$$\frac{x_j^3 e^{-\frac{x_j}{\theta}}}{\Gamma(4)\theta^4} \qquad x_j > 0$$

Thus

$$\begin{aligned} L(\theta) &= \left(\frac{1}{\Gamma(4)\theta^4}\right) x_1^3 e^{-x_1/\theta} \cdot \left(\frac{1}{\Gamma(4)\theta^4}\right) x_2^3 e^{-x_2/\theta} \cdots \left(\frac{1}{\Gamma(4)\theta^4}\right) x_n^3 e^{-x_n/\theta} \\ &= \frac{1}{(\Gamma(4)\theta^4)^n} \left(\prod_{j=1}^n x_j\right)^3 e^{-\frac{1}{\theta}\sum_{j=1}^n x_j} \\ &= \frac{1}{(\Gamma(4)\theta^4)^n} \left(\prod x_j\right)^3 e^{-\frac{1}{\theta}\sum x_j} \end{aligned}$$

(ii) We find that $L(\theta)$ can be written in the form given in the factorization theorem,

$$L(\theta) = g\left(\vec{S}\left(\vec{x}\right); \theta\right) h\left(\vec{x}\right)$$

with

$$\begin{aligned} d &= 1 \text{ (so } S \text{ is 1-dimensional)} \\ h\left(\vec{x}\right) &= \left(\prod x_j\right)^3 \\ S\left(\vec{x}\right) &= \sum x_j \\ g\left(S; \vec{\theta}\right) &= \frac{1}{(\Gamma(4)\theta^4)^n} e^{-\frac{1}{\theta}S} \end{aligned}$$

Hence $S\left(\vec{x}\right) = \sum x_j$ is a sufficient statistic.

(b) (Rossi 4.3.6 solution)

(i) Observe that the pdf of each random variable X_j is given by

$$f(x;\theta) = \frac{x^3 e^{-\frac{x}{\theta}}}{\Gamma(4)\theta^4} = \frac{1}{\Gamma(4)\theta^4} x^3 e^{-\frac{1}{\theta}x}$$

So that $f(x;\theta)$ has the form, $c(\theta)h(x)e^{q(\theta)t(x)}$, of a one-parameter exponential family. Here, $c(\theta) = \frac{1}{\Gamma(4)\theta^4}, h(x) = x^3, q(\theta) = -1/\theta$, and $t(x) = x$. It follows from Theorem 4.15 that $T = \sum X_j$ is a complete minimal sufficient statistic for θ.

(ii)

$$
\begin{aligned}
E\left[\left(\frac{\partial}{\partial\theta}\ln\left(\frac{X^3e^{-\frac{X}{\theta}}}{\Gamma(4)\theta^4}\right)\right)^2\right] &= -E\left[\frac{\partial^2}{\partial\theta^2}\ln\left(\frac{X^3e^{-\frac{X}{\theta}}}{\Gamma(4)\theta^4}\right)\right] \\
&= -E\left[\frac{\partial^2}{\partial\theta^2}\left(3\ln(X) - \frac{X}{\theta} - \ln\Gamma(4) - 4\ln\theta\right)\right] \\
&= -E\left[-\frac{2X}{\theta^3} + \frac{4}{\theta^2}\right] \\
&= \frac{2(4\theta)}{\theta^3} + \frac{4}{\theta^2} \\
&= \frac{4}{\theta^2}
\end{aligned}
$$

Thus, Fisher's information number for θ is given by

$$I_n(\theta) = nI(\theta) = \frac{4n}{\theta^2}$$

(iii) Using the expression for $I_n(\theta)$ we derived in the previous part, we find that

$$\mathrm{CRLB}_\theta = \frac{1}{\frac{4n}{\theta^2}} = \frac{\theta^2}{4n}$$

(iv) First note that,

$$E\left[\sum X_j\right] = 4n\theta$$

Therefore

$$E\left[\frac{1}{4n}\sum X_j\right] = \theta$$

Thus the estimator $T = \frac{1}{4n}\sum X_j = \frac{1}{4n}S$ is unbiased and is a function of S alone; therefore, by Theorem 4.16, T is an UMVUE for θ.