Stochastic Processes November 20, 2023
Sharif University of Technology CE 40-695
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Time: 20 mins

Name: Std. Number:

Quiz 6

Questions

1. Let X takes on the specific values of $1, \ldots, K$ with probabilities $\theta_1, \ldots, \theta_K$ respectively, where, $\forall i, \theta_i > 0, \sum_{i=1}^K \theta_i = 1$. Suppose that $X_1, \ldots, X_n$ are iid samples from the same distribution as X. Suppose that $\theta = (\theta_1, \ldots, \theta_K)$ is unknown. Find a sufficient statistic for θ.

2. Suppose X is a random variable. We know that the probability that X be greater than some number x is given by,
$$\Pr(X > x) = \begin{cases} \left(\frac{\theta}{x}\right)^\alpha & x > \theta \\ 1 & \text{o.w.} \end{cases}$$
where, (α, θ) are positive parameters. Let $X_1, \ldots, X_n$ be iid samples of the variable X. Find a two dimensional sufficient statistic for (α, θ).

Solution:

1. This is the typical multinomial distribution. We first identify the joint pdf of $X_1, \ldots, X_n$. For simplicity we use the one-of-K coding. i.e. each X_i is a K dimensional vector with one entry equal to one. So,
$$f\left(X_1, \ldots, X_n \mid \theta\right) = \prod_{i=1}^n \prod_{k=1}^K \theta_k^{X_i(k)} = \prod_{i=1}^K \theta_k^{\sum_{i=1}^n X_i(k)}$$
Let N_k be the number of X_i s which equal k. By definition we have $N_k = \sum_{j=1}^K X_i(j)$. So, by the factorization theorem $T(X) = (N_1, \ldots, N_K)$ is a sufficient statistic for $\theta = (\theta_1, \ldots, \theta_K)$

2. We first should identify the pdf of random variable X. It is straight forward to show that the pdf is as the following:
$$f(x \mid \theta, \alpha) = \frac{\alpha \theta^\alpha}{x^{\alpha+1}} \mathcal{I}_{(\theta, \infty)}(x)$$
Then, the joint pdf is
$$f\left(X_1, \ldots, X_n \mid \theta, \alpha\right) = \frac{\left(\alpha \theta^\alpha\right)^n}{\left(\prod_{i=1}^n x_i\right)^{\alpha+1}} \mathcal{I}_{(\theta, \infty)}\left(x_{(1)}\right)$$
Thus, by the factorization theorem, $\left(\prod_{i=1}^n x_i, x_{(1)}\right)$ is a sufficient statistic for (α, θ)