In the name of GOD.

Sharif University of Technology

Stochastic Process

Fall 2023

Hamid R. Rabiee

Quiz 5 - solution	Point process + Gaussian Process	Release : 7 Nov

- Brownian motion is a Gaussian process W_t such that $E[W_t] = 0$ and $cov[W_t, W_s] = min(t, s)$.
- If Z is a standard normal random variable then we have : $P(Z \leq x) = \phi(x)$

1. Harry is playing some game. At each given time moment t the number of points he gets is determined by the value of the process $X_t = \sigma W_t$, that is, he earns points if $X_t > 0$ and loses if $X_t < 0$. Every hour (at time moments t = 1,2, . . .) the results are automatically recorded, and if the sum of 3 results in a row is greater than 10, Harry gets a special prize. Find the probability that Harry gets this prize after 3 hours of playing.(70 points)
   Note : The answer should be in the form of ϕ

   Solution :
   We need to find the probability that :
   $P(X_1 + X_2 + X_3 > 10) = P(\sigma(W_1 + W_2 + W_3) > 10) = 1 - p(W_1 + W_2 + W_3 \leq \frac{10}{\sigma})$
   As
   $E[W_1 + W_2 + W_3] = E[W_1] + E[W_2] + E[W_3] = 0$
   and
   $Var(W_1 + W_2 + W_3) = Var(W_1) + Var(W_2) + Var(W_3) + 2cov(W_1, W_2) + 2cov(W_2, W_3)$
   $+2cov(W_1, W_3) = 1 + 2 + 3 + 1 * 2 + 2 * 2 + 1 * 2 = 14$

   So we have :

$$1 - P(W_1 + W_2 + W_3 \leq \frac{10}{\sigma}) = 1 - \phi(\frac{10}{\sigma\sqrt{14}})$$

2. Let $N(t), t \in [0, \infty)$ be a Poisson process with rate λ and X_1 be its first arrival time. show : (30 points)

$$P(X_1 \leq x | N(t) = 1) = \frac{x}{t} \text{ for } 0 \leq x \leq t$$

   Solution:
   For $0 \leq x \leq t$, we can write :
   $P(X_1 \leq x | N(t) = 1) = \frac{P(X_1 \leq x, N(t)=1)}{P(N(t)=1)}$
   We know that :
   $P(N(t) = 1) = \lambda t e^{-\lambda t}$

$P(X_1 \leq x, N(t) = 1) = P($ one arrival in $(0, x]$ and no arrivals in $(x, t])$
$= [\lambda x e^{-\lambda x}][e^{-\lambda(t-x)}]$

$P(X_1 \leq x | N(t) = 1) = \frac{x}{t}$, for $0 \leq x \leq t$