Time: 20 mins

Name: Std. Number:

Quiz 3 (Stochastic Analysis of Systems, and Ergodicity)

Questions

1. Given that $X(t)$ is a WSS Gaussian process with mean 0 and autocovariance function $R_x(\tau) = e^{-|\tau|}$, and A is a normal random variable with mean 0 and standard deviation 1, independent of X, is the process $Y(t) = X(t) + A$ a mean-ergodic process? Please provide an explanation for your answer.

2. Consider the linear time-invariant system defined by the difference equation:

$$Y(n) = \alpha Y(n-1) + (1-\alpha)W(n)$$

where $W(n)$ is a wide-sense stationary (WSS) process.

- Determine the impulse response $h(n)$ of the system.
- Compute the expected value $E[Y(n)]$ and the autocovariance function $R_{yy}(n, s)$.

Stochastic Processes
Sharif University of Technology
Hamid R. Rabiee

CE 40-695

Quiz 3 Solution (Stochastic Analysis of LTI Systems)

Solutions

1. **Impulse Response $h(n)$:**

   Given a system with a difference equation and an impulse input $\delta(n)$, the output is the impulse response $h(n)$ of the system.

$$h(n) = \alpha h(n-1) + (1-\alpha)\delta(n) \tag{1}$$

For $n = 0$:

$$h(0) = \alpha h(-1) + (1-\alpha)\delta(0) \tag{2}$$

Given that $h(-1) = 0$ (because the system hasn't started yet) and $\delta(0) = 1$ (by definition of the impulse function), we have:

$$h(0) = 1 - \alpha \tag{3}$$

Solving for $h(n)$ yields:

$$h(n) = (1-\alpha)\alpha^n U(n) \tag{4}$$

where $U(n)$ is the unit step function.

Expected Value $E[Y(n)]$:

Given $W(n)$ is a WSS process, its mean will be constant. Thus, using the linearity property, the expected value $E[Y(n)]$ is given by the convolution of the impulse response $h(n)$ with the expected value of the input:

$$E[Y(n)] = \sum_{k=0}^{\infty} h(k)E[W(n-k)] \tag{5}$$

Assuming $E[W(n)] = \mu_w$, a constant, the equation becomes:

$$E[Y(n)] = \mu_w \sum_{k=0}^{\infty} h(k) \tag{6}$$

Which further simplifies to:

$$E[Y(n)] = \mu_w \tag{7}$$

Autocovariance Function $R_{yy}(n, s)$:

The autocovariance function is given by:

$$R_{yy}(m) = E[Y(n)Y(n+m)] \tag{8}$$

Breaking it down based on the given system:

$$R_{yy}(m) = \alpha E[Y(n-1)Y(n+m)] + (1-\alpha)E[W(n)Y(n+m)] \tag{9}$$

If $W(n)$ and $Y(n)$ are jointly WSS, the equation can be simplified:

$$R_{yy}(m) = \alpha R_{yy}(m-1) + (1-\alpha)R_{wy}(m) \tag{10}$$

Where $R_{wy}(m)$ represents the cross-covariance between $W(n)$ and $Y(n)$.

Given that $Y(n)$ also satisfies the properties of WSS, we conclude that $Y(n)$ is Wide Sense Stationary.