Stochastic Processes September 26, 2023
Sharif University of Technology CE 40-695
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Time: 20 mins

Name: Std. Number:

Prerequisite Quiz

Questions

1. Suppose that m and n are positive integers. What is the probability that a randomly chosen positive integer less than mn is not divisible by either m or n?(10 points)

2. A simple example of a random variable is the *indicator* of an event A, which is denoted by I_A:

$$I_A(\omega) = \begin{cases} 1, & if\ \omega \in A \\ \\ 0, & Otherwise. \end{cases}$$

   (a) Prove that two events A and B are independent if and only if the associated indicator random variables, I_A and I_B are independent.(6 points)

   (b) If $X = I_A$, find $E[X]$. (4 points)

3. Let X be a continuous random variable with the following PDF

$$F_X(x) = \begin{cases} ce^{-x}, & x > 0 \\ \\ 0, & Otherwise. \end{cases}$$

   where c is a positive constant.

   (a) Find c

   (b) Find the CDF of X, $F_X(x)$

   (c) Find $P(1 < X < 3)$

Solutions:

1) We have $mn-1$ integers less than mn. Furthermore, there are $n-1$ multiples of m and $m-1$ multiples of n less than mn; However, $\frac{mn}{lcm(m,n)}-1$ of these multiples are the same. Hence by the Inclusion-Exclusion law we have $m+n-1-gcd(m,n)$ multiples of either m or n. By doing some algebra, the probability of not choosing a multiple of either one would be:

$$\frac{(m-1)(n-1)+gcd(m,n)-1}{mn-1}$$

2)

(a) We know that I_A is a random variable that maps a 1 to the real number line if ω occurs within an event A and maps a 0 to the real line if ω occurs outside of event A. A similar argument holds for event B. Thus we have,

$$I_A(\omega) = \begin{cases} 1, & \textit{with probability } P(A) \\ 0, & \textit{with probability } 1-P(A) \end{cases}$$

If the random variables, A and B, are independent, we have $P(A\cap B) = P(A)P(B)$. The indicator random variables, I_A and I_B, are independent if, $P_{I_A,I_B}(x,y) = P_{I_A}(x)P_{I_B}(y)$. First suppose I_A and I_B are independent, we have:

$$P(A\cap B) = P_{I_A,I_B}(1,1) = P_{I_A}(1)P_{I_B}(1) = P(A)P(B)$$

now suppose A and B are independent,

$$P_{I_A,I_B}(1,1) = P(A\cap B) = P(A)P(B) = P_{I_A}(1)P_{I_B}(1)$$

$$P_{I_A,I_B}(0,1) = P(A^c\cap B) = P(A^c)P(B) = P_{I_A}(0)P_{I_B}(1)$$

$$P_{I_A,I_B}(1,0) = P(A\cap B^c) = P(A)P(B^c) = P_{I_A}(1)P_{I_B}(0)$$

$$P_{I_A,I_B}(1,1) = P(A\cap B) = P(A^c)P(B^c) = P_{I_A}(0)P_{I_B}(0)$$

(b) If $X = I_A$, we know that

$$E[X] = E[I_A] = 1.P(A) + 0.(1-P(A)) = P(A)$$

3)a. To find c, we can use Property 2 above, in particular

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} f_X(u)du \\ &= \int_0^{\infty} ce^{-u}du \\ &= c\Big[-e^{-x}\Big]_0^{\infty} \\ &= c. \end{aligned}$$

Thus, we must have $c = 1$.

b. To find the CDF of X, we use $F_X(x) = \int_{-\infty}^{x} f_X(u)du$, so for $x < 0$, we obtain $F_X(x) = 0$. For $x \geq 0$, we have

$$F_X(x) = \int_0^x e^{-u}du = 1 - e^{-x}.$$

Thus,

$$F_X(x) = \begin{cases} 1 - e^{-x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

c. We can find $P(1 < X < 3)$ using either the CDF or the PDF. If we use the CDF, we have

$$P(1 < X < 3) = F_X(3) - F_X(1) = \Big[1 - e^{-3}\Big] - \Big[1 - e^{-1}\Big] = e^{-1} - e^{-3}.$$

Equivalently, we can use the PDF. We have

$$\begin{aligned} P(1 < X < 3) &= \int_1^3 f_X(t)dt \\ &= \int_1^3 e^{-t}dt \\ &= e^{-1} - e^{-3}. \end{aligned}$$