In the name of GOD.

Sharif University of Technology

Stochastic Process

Fall 2023

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Practical Homework 2	Estimation Theory	Deadline : 3 Dey

1. let $\mathbf{Y}_1$, ... ,$\mathbf{Y}_n$ be a random sample from an exponential distribution with mean θ, and consider the following estimators for θ:

   - $\hat{\theta} = \bar{\mathbf{Y}}$
   - $\tilde{\theta} = \mathbf{n}\,\min(\mathbf{Y}_i)$
   - $\dot{\theta} = \mathbf{S}$, the sample standard deviation

   Through simulating 100,000 random samples of size $\mathbf{n} = 10$ from an exponential distribution with mean $\theta = 2$ compute and compare the bias and MSEs of All three estimator. Compare your simulation results with the analytical results . Also for estimators that are unbiased, calculate the relative efficiency.

2. the data set *ex*0428, "Darwin's Data", in the library *Sleuth*2 of R project has following description:

   > Plant heights (inches) for 15 pairs of plants of the same age, one of which was grown from a seed from a cross-fertilized flower and the other of which was grown from a seed from a self-fertilized flower.

   Suppose we want to model the difference μ in heights of cross and self-fertilized flowers, using a normal density. We'll also suppose that the variance of differences in heights to be $\sigma^2 = 20$. Use a normal prior for μ, with mean zero and large variance, say $\delta^2 = 100$.

   (a) Find the Bayes estimator for the difference μ in height.

   (b) Make a plot of the posterior distribution. Would you say there is evidence that the true difference μ is lager than zero?

3. Suppose $\mathbf{Y}_1$, ... ,$\mathbf{Y}_n$ are independent random variable with distribution

   $$f_\theta(\mathrm{y}) = \theta\,\mathrm{y}^{-(\theta+1)} \quad for\ \mathrm{y} > 1,$$

   where $\theta > 0$.

   (a) Compute the maximum likelihood and the Fisher information $\mathbf{I}(\theta)$.

   (b) For MLE, $\hat{\theta}$, we have $\mathbf{V}(\hat{\theta}) \approx \mathbf{I}(\theta)^{-1}$, it can be shown that we have "asymptotic normality" of the estimator; in fact:

   $$\hat{\theta} \approx \mathcal{N}(\theta, \mathbf{I}(\theta)^{-1})$$

This allows for hypothesis testing about θ and confidence intervals. The approximate test using the information to approximate the variance of the estimator is called a **Wald test**. Use a Wald test to determine an approximation $p-$value for $H_0 : \theta = 4$ versus $H_a : \theta > 4$, using the following random sample from the distribution f_θ(y):

$$1.53,\ 1.36,\ 3.64,\ 1.18,\ 1.31,\ 1.59,\ 1.89,\ 1.47,\ 1.25,\ 2.98$$

(c) simulate the null distribution of the MLE from (a), using one million loops. This will give a $p-$value that is precise to about three decimal places. Compare this to the Wald test $p-$value, which is based on an approximation of the null distribution, Finally, make a histogram of the true null hypothesis density of your test statistic, and superimpose the approximation normal density to see the reason for the discrepancy of $p-$value.