In the name of GOD.

Stochastic Process

Sharif University of Technology

Fall 2023

Hamid R. Rabiee

Homework 6 | Sampling | Deadline : 19 Dey

1. Use rejection sampling to estimate Area Under the Curve for any bounded and positive function $f(x)$ in the interval $0 \leq x \leq 1$. Describe your method.
   - Use your method to estimate AUC for e^x.
   - Increase number of samples and observe how the result changes compared to its actual value.
   - Find the minimum number of samples to achive 10^{-3} accuracy with probability 99%.
2. Suppose city populations are drawn from the following distribution.

$$f(x; \alpha, c) = \frac{\alpha c^{\alpha}}{x^{\alpha+1}} \ , \ x \geq c$$

   Which is also known as the Pareto distribution.
   - Derive $\mathbb{P}[\alpha \mid c; x_{1\ldots n}]$ and $\mathbb{P}[c \mid \alpha; x_{1\ldots n}]$.
   - Use gibbs sampling to estimate α and c for the following data. Describe your initial values for α and c.

$$730000, 403000, 269000, 228000, 229000$$

3. Given Bernoulli random variables $X_{1\ldots k}$ and $Y_{1\ldots m}$ denote the n-th iteration of gibbs sampling as $X_1^{(n)}, \ldots, X_k^{(n)}, Y_1^{(n)}, \ldots, Y_m^{(n)}$. Prove that

$$\mathbb{P}[X_1 = x_1, \ldots, X_k = x_k | Y_1 = y_1, \ldots, Y_m = y_m] = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \prod_{j=1}^{k} \mathbf{1}[X_j^{(i)} = x_j]$$

   Where $\mathbf{1}[C] = 1$ if the C condition holds.
4. In many practical applications for Metropolis-Hastings algorithms, we tend to forget about the the first 10% of the data also known as *burn-in*. What are the main reasons?
5. Suppose in a given Poisson Process the rate suddenly changes.

$$X_1, \ldots, X_k \sim PP(\mu) \ , \ X_{k+1}, \ldots, X_m \sim PP(\lambda)$$

   Where

$$\mu \sim \text{Gamma}(\alpha_1, \beta_1) \ , \ \lambda \sim \text{Gamma}(\alpha_2, \beta_2)$$

   - Find a non-informative and applicable prior for k.

- Suppose that priors for μ, λ, k are independent. Derive $\mathbb{P}[k|\mu, \lambda; X_{1\ldots n}]$, $\mathbb{P}[\mu|k, \lambda; X_{1\ldots n}]$ and $\mathbb{P}[\lambda|\mu, k; X_{1\ldots n}]$ and give update rules for each iteration of gibbs sampling.

6. Let $\mathbb{P}[\theta] = \frac{1}{\pi(1+\theta^2)}$ and $\mathbb{P}[X|\theta] = \mathcal{N}(\theta, 1)$. Use rejection sampling to describe a method to sample from the distribution $\mathbb{P}[\theta|x]$. Explain why we have to use sampling in this case.