In the name of GOD.

Stochastic Process

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Homework 4 | Estimation Theory | Deadline : 14 Azar

1. Let $X_1, \ldots, X_n$ be iid with pdf

$$f(x \mid \theta) = \frac{1}{\theta}, \quad 0 \leq x \leq \theta, \quad \theta > 0.$$

Estimate θ using both the method of moments and maximum likelihood. Calculate the means and variances of the two estimators. Which one should be preferred and why?

Solution:
This is a uniform $(0, \theta)$ model. We denote $\tilde{\theta}$ for the method of moments estimator and $\hat{\theta}$ for MLE estimator. This is a uniform $(0, \theta)$ model. So $\mathbb{E}[X] = (0 + \theta)/2 = \theta/2$. The method of moments estimator is the solution to the equation $\tilde{\theta}/2 = \bar{X}$, that is, $\tilde{\theta} = 2\bar{X}$. Because $\tilde{\theta}$ is a simple function of the sample mean, its mean and variance are easy to calculate. We have

$$\mathbb{E}[\tilde{\theta}] = 2\mathbb{E}[\bar{X}] = 2\mathbb{E}[X] = 2\frac{\theta}{2} = \theta, \quad \text{and} \quad \operatorname{Var}[\tilde{\theta}] = 4\operatorname{Var}[\bar{X}] = 4\frac{\theta^2/12}{n} = \frac{\theta^2}{3n}$$

The likelihood function is

$$L(\theta \mid \mathbf{x}) = \prod_{i=1}^{n} \frac{1}{\theta} I_{[0,\theta]}\left(x_i\right) = \frac{1}{\theta^n} I_{[0,\theta]}\left(x_{(n)}\right) I_{[0,\infty)}\left(x_{(1)}\right),$$

where $x_{(1)}$ and $x_{(n)}$ are the smallest and largest order statistics. For $\theta \geq x_{(n)}, L = 1/\theta^n$, a decreasing function. So for $\theta \geq x_{(n)}, L$ is maximized at $\hat{\theta} = x_{(n)}. L = 0$ for $\theta < x_{(n)}$. So the overall maximum, the MLE, is $\hat{\theta} = X_{(n)}$. The pdf of $\hat{\theta} = X_{(n)}$ is $nx^{n-1}/\theta^n, 0 \leq x \leq \theta$. This can be used to calculate

$$\mathbb{E}[\hat{\theta}] = \frac{n}{n+1}\theta, \quad \mathbb{E}[\hat{\theta}^2] = \frac{n}{n+2}\theta^2 \quad \text{and} \quad \operatorname{Var}[\hat{\theta}] = \frac{n\theta^2}{(n+2)(n+1)^2}.$$

$\tilde{\theta}$ is an unbiased estimator of $\theta; \hat{\theta}$ is a biased estimator. If n is large, the bias is not large because $n/(n+1)$ is close to one. But if n is small, the bias is quite large. On the other hand, $\operatorname{Var}[\hat{\theta}] < \operatorname{Var}[\tilde{\theta}]$ for all θ. So, if n is large, $\hat{\theta}$ is probably preferable to $\tilde{\theta}$.

2. $X_1, X_2, ..., X_n$ are iid samples from $\mathcal{N}(\mu, \sigma^2)$. Consider we know σ^2. Compute UMVUE for μ^3.

Solution:

For MLE we have:

$$L(\theta|\{X_i\}) = (\frac{1}{\sigma\sqrt{2\pi}})^n \exp\left(\frac{-1}{2}\sum_{i=1}^{n}(\frac{X_i-\mu}{\sigma})^2\right)$$
$$\to \log L(\theta|\{X_i\}) = -n\log(\sigma\sqrt{2\pi}) - \frac{1}{2}\sum_{i=1}^{n}(\frac{X_i-\mu}{\sigma})^2$$
$$\frac{\partial \log L}{\partial \mu} = \frac{1}{\sigma^2}\left(-n\mu + \sum_{i=1}^{n} X_i\right) = 0 \to \hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

We denote $\theta = \mu^3$. From the invariance property we know:

$$\hat{\theta} = \frac{1}{n^3}\left(\sum_{i=1}^{n} X_i\right)^3$$

To check bias of this estimator we must compute expected value:

$$\mathbb{E}[\hat{\theta}] = \frac{1}{n^3}\mathbb{E}\left[\left(\sum_{i=1}^{n} X_i\right)^3\right]$$
$$= \mu^3 + \frac{3\sigma^2\mu}{n}$$

From Rao-Blackwell we have:

$$\left.\begin{array}{r}\Rightarrow \hat{\theta}_{\text{unbiased}} = \hat{\theta}_{MLE} - \frac{3\sigma^2\mu}{n} \\ \bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \Rightarrow E[\bar{X}] = \mu\end{array}\right\} \Rightarrow \hat{\theta}_{\text{unbiased}} = (\bar{X})^3 - \frac{3\sigma^2}{n}\bar{X}$$
$$\hat{\theta}(x) = \left\{\begin{array}{l} T(x) = \bar{X} \\ g(T) = T^3 - \frac{3\sigma^2}{n}T \end{array}\right\} \Rightarrow \bar{X} \text{ is ss for } \mu^3$$
$$\hat{\theta}_{\text{UMVUE}} = \mathbb{E}\left[(\bar{X})^3 - \frac{3\sigma^2}{n}\bar{X} \mid \bar{x}\right] = (\bar{X})^3 - \frac{3\sigma^2}{n}\bar{X}$$

3. Let $X_1, \ldots, X_n$ be iid with one of two pdfs. If $\theta = 0$, then

$$f(x \mid \theta) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

while if $\theta = 1$, then

$$f(x \mid \theta) = \begin{cases} 1/(2\sqrt{x}) & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the MLE of θ.

Solution:
$L(0 \mid \mathbf{x}) = 1, 0 < x_i < 1$, and $L(1 \mid \mathbf{x}) = \prod_i 1/\left(2\sqrt{x_i}\right), 0 < x_i < 1$. Thus, the MLE is 0 if $1 \geq \prod_i 1/\left(2\sqrt{x_i}\right)$, and the MLE is 1 if $1 < \prod_i 1/\left(2\sqrt{x_i}\right)$.

4. $X_1, X_2, ..., X_n$ are iid samples from a distribution with PDF as follows:

$$f_X(x) = \frac{1}{2\theta} \exp(-\frac{|x|}{\theta}) \text{ where } \theta > 0$$

Find the MSE of the following estimator:

$$\hat{\theta} = \frac{1}{n} \sum_{i=1}^{n} |X_i|$$

Solution:

$$E[\hat{\theta}] = E[|x|] = \int_{-\infty}^{\infty} |x| \frac{1}{2\theta} e^{-\frac{|x|}{\theta}} dx = \frac{1}{\theta} \int_{0}^{\infty} x e^{-\frac{x}{\theta}} dx = \theta$$

$$\operatorname{Var}[\hat{\theta}] = \frac{1}{n} \operatorname{Var}[|X|]$$

$$\operatorname{Var}[|X|] = E\left[|X|^2\right] - E[|X|]^2 = E\left[|X|^2\right] - \theta^2$$

$$E\left[X^2\right] = \int_{-\infty} x^2 \frac{1}{2\theta} e^{-\frac{|x|}{\theta}} dx = \frac{1}{\theta} \int_{0} x^2 e^{-\frac{x}{\theta}} dx = 2\theta^2$$

$$\operatorname{Var}[|X|] = \theta^2 \rightarrow \operatorname{Var}[\hat{\theta}] = \frac{\theta^2}{n}$$

$$\operatorname{MSE}(\hat{\theta}) = (E[\hat{\theta}] - \theta)^2 + \operatorname{Var}[\hat{\theta}] = \frac{\theta^2}{n}$$

5. $X_1, X_2, ..., X_n$ are iid samples from a distribution with PDF as follows:

$$f(x \mid \theta) = \frac{\theta}{(1+x)^{\theta+1}}, \quad 0 < \theta < \infty, 0 < x < \infty$$

Find a sufficient statistics for θ.

Solution:
First note that

$$\begin{aligned} f\left(x_1, x_2, \ldots, x_n \mid \theta\right) &= \prod_{i=1}^{n} \frac{\theta}{\left(1+x_i\right)^{\theta+1}} \\ &= \frac{\theta^n}{\left[\prod_{i=1}^{n}\left(1+x_i\right)\right]^{\theta+1}} \\ &= \frac{\theta^n}{u^{\theta+1}} \end{aligned}$$

Here we have

$$\begin{aligned} U &= \prod_{i=1}^{n}\left(1+X_i\right) \\ g(u, \theta) &= \frac{\theta^n}{u^{\theta+1}} \\ h\left(x_1, x_2, \ldots, x_n\right) &= 1 \end{aligned}$$

Thus, by the Factorization Theorem,

$$U = \prod_{i=1}^{n}\left(1+X_i\right)$$

is a sufficient statistic for θ.

6. We have a random variable of X with discrete distribution of:

$$p_X(x|\theta) = \begin{cases} \frac{1}{2\theta+1} & \text{if } x \in \{-\theta, -\theta+1, \cdots, \theta-1, \theta\} \\ 0 & \text{otherwise} \end{cases}$$

By considering $\theta \in \mathbb{N}$, find a sufficient statistic using factorization theorem.

Solution:
For the joint distribution we have:

$$p_X(X|\theta) = \begin{cases} \frac{1}{(2\theta+1)^n} & \text{if } \forall_i X_i \in \{-\theta, -\theta+1, \cdots, \theta-1, \theta\} \\ 0 & \text{otherwise} \end{cases}$$

We can rewrite the condition as:

$$p_X(X|\theta) = \begin{cases} \frac{1}{(2\theta+1)^n} & \text{if } \max|X_i| \leq \theta \\ 0 & \text{otherwise} \end{cases}$$

We can consider $h(x) = 1$ and using factorization theorem we can conclude that $\max|X_i|$ is a sufficient statistic.