In the name of GOD.

Stochastic Process

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Homework 3 — Stochastic Processes and Stationary Stochastic Processes — Deadline : 21 Aban

1. Find the autocorrelation or PSD of the following processes.

   a) The PSD of a noise process is given by :

$$S_N(\omega) = \frac{N_0}{2} \quad if \quad |\omega \pm \omega_c| \leq \frac{W}{2}$$

$$0 \quad otherwise$$

   Find the autocorrelation of the process.

   b) The autocorrelation function of a WSS process X(t) is given by :

$$R_X(\tau) = a^2 e^{-b|\tau|} \quad b > 0$$

   Find the power spectral density of the process.

2. Suppose that passengers arrive at a train station in accordance with a Poisson process $N(t) : t \geq 0$ having rate λ. If the time between two consecutive trains is one hour, compute the expected total (combined) waiting time of all the passengers, assuming that all the passengers waiting in the station board the train when it arrives.

3. Prove the following :

   a) If $X(t)$ is a WSS random process with psd $S_x(\omega)$, then the psd of the derivative of $X(t)$ is equal to :

$$S_{X'}(\omega) = \omega^2 S_X(\omega)$$

   b) If $(N_t^1)_{t>0}$ and $(N_t^2)_{t\geq 0}$ are two independent Poisson processes with intensities λ_1 and λ_2 then $(N_t^1 + N_t^2)_{t\geq 0}$ is a Poisson process with intensity $\lambda_1 + \lambda_2$.

4. Customers arrive at an ATM machine in accordance with a Poisson process with rate 15 per hour. The amount of money withdrawn on each transaction (assuming there is no failed transaction) is an i.i.d random variable with mean 20 and standard deviation 50. Suppose that the machine is in use 18 hours per day :

   a) Calculate the expected amount of the total daily withdraw.

   b) Compute the variance of the total daily withdraw.

5. Vehicles stopping at a roadside restaurant form a Poisson process $N(t)$ with rate $\lambda = 20$ vehicles per hour. A vehicle has 1,2,3,4 or 5 persons in it with respective probabilities 0.3,0.3,0.2,0.1 and 0.1. We would like to find the expected number of persons arriving at the restaurant within one hour.

6. Let $X_1 \sim N(0, \sigma_1^2)$ and let $X_2 \sim N(0, \sigma_2)$ be independent of X_1. Convolve the density of X_1 with X_2 to show that $X_1 + X_2$ is Gaussian, $N(0, \sigma_1^2 + \sigma_2^2)$

7. Let ξ be a Gaussian $N(0,1)$ random variable. Let $x > 0$.

   a) Prove that $\frac{1}{(2\pi)^{\frac{1}{2}}}(\frac{1}{x} - \frac{1}{x^3})e^{-\frac{x^2}{2}} \leq P(\xi > x) \leq \frac{1}{(2\pi)^{\frac{1}{2}}}\frac{1}{x}e^{-\frac{x^2}{2}}$

   b) Prove that $P(\xi > x) \leq e^{-\frac{x^2}{2}}$