In the name of GOD.

Sharif University of Technology

Stochastic Process

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Homework 3	Stochastic Processes and Stationary Stochastic Processes	Deadline : 21 Aban

1. - The distribution of stars in the universe (assuming it remains constant over time).
   - Daily temperature readings in a specific location over a year (assuming the mean and variance remain constant).
   - The growth of a child's height over time (since the mean and variance change as the child grows).
   - No, a single sample path is not enough to determine if a process is WSS. Multiple sample paths or statistical properties of the process are needed.
   - No, being SSS does not imply that the process is independent and identically distributed.

2. - $\sin(\tau)$: Not valid. The autocorrelation function must be even, and $\sin(\tau)$ is odd.
   - $e^{-\tau^2}$:
     - Symmetry: The function $e^{-\tau^2}$ is even, meaning $e^{-\tau^2} = e^{-(-\tau)^2}$. It satisfies the symmetry requirement.
     - Non-negative Definite: This function is always non-negative for any value of τ, and it satisfies the non-negative definite requirement. Hence, $e^{-\tau^2}$ can serve as a valid autocorrelation function.

3.

$$\begin{aligned} R_y(\tau) &= E[y(t) \cdot y(t+\tau)] \\ &= E[(x(t+a) - x(t-a)) \cdot (x(t+\tau+a) - x(t+\tau-a))] \\ &= E[x(t+a) \cdot x(t+\tau+a)] - E[x(t+a) \cdot x(t+\tau-a)] - \\ & E[x(t-a) \cdot x(t+\tau+a)] + E[x(t-a) \cdot x(t+\tau-a)] \\ &= 2R_x(\tau) - R_x(\tau+2a) - R_x(\tau-2a) \end{aligned} \tag{1}$$

The power spectral density is the Fourier transform of $R_y(\tau)$:

$$S_y(\omega) = \mathcal{F}\{2R_x(\tau) - R_x(\tau+2a) - R_x(\tau-2a)\} \tag{2}$$

Using Fourier properties:

$$\begin{aligned} S_y(\omega) &= 2S_x(\omega) - e^{j\omega 2a} S_x(\omega) - e^{-j\omega 2a} S_x(\omega) \\ &= [2 - e^{j\omega 2a} - e^{-j\omega 2a}] S_x(\omega) \\ &= [2 - 2\cos(2a\omega)] S_x(\omega) \\ &= 2[1 - \cos(2a\omega)] S_x(\omega) \\ &= 4sin^2(a\omega) S_x(\omega) \end{aligned} \tag{3}$$

4. Consider a simple random walk on the integers. At each time step, the process either moves one step to the right with probability p or one step to the left with probability $1 - p$.

   - Ergodicity: Over a long period, the random walk will visit all states (integers) infinitely often. The recurrence of the random walk ensures that the time average reflects the behavior across all states. This leads the time average to match the ensemble average, confirming the process as ergodic.
   - Stationarity: The mean of the random walk is $E[X_t] = 2pt - t$, which depends on time t. Therefore, the process is not stationary.

5.

$$H(\omega) = \mathcal{F}\{h(t)\} = \frac{i}{\sqrt{2\pi}(2i + \omega)} \tag{4}$$

$$\begin{aligned} S_Y(\omega) &= |H(\omega)|^2 \times S_X(\omega) \\ &= \left| \frac{i}{\sqrt{2\pi}(2i + \omega)} \right|^2 \times \frac{1}{\omega^2 + 4} \\ &= \frac{1}{2\pi(4 + \omega^2)|2I + \omega|^2} \end{aligned} \tag{5}$$

6.

$$\begin{aligned} E[X(t)] &= \int_0^1 u(t - v)\,dv \\ &= \begin{cases} 1 & \text{for } t \geq 1 \\ t & \text{for } 0 \leq t < 1 \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{6}$$

$$\begin{aligned} E[Y(t)] &= \int_0^1 \delta(t - v)\,dv \\ &= \begin{cases} 1, & 0 \leq t \leq 1 \\ 0, & \text{otherwise} \end{cases} \end{aligned} \tag{7}$$

$$\begin{aligned} R_{XX}(t_1, t_2) &= E[X(t_1)X(t_2)] \\ &= \int_0^1 u(t_1 - v)u(t_2 - v)\,dv \\ &= \begin{cases} 1, & t_1 \geq 1 \text{ and } t_2 \geq 1 \\ min(t_1, t_2), & 0 \leq min(t_1, t_2) < 1 \\ 0, & \text{otherwise} \end{cases} \end{aligned} \tag{8}$$

$$
\begin{aligned}
R_{YY}(t_1, t_2) &= E[Y(t_1)Y(t_2)] \\
&= E[\delta(t_1 - v)\delta(t_2 - v)] \\
&= \begin{cases} 1 & \text{if } t_1 = t_2 \\ 0 & \text{otherwise} \end{cases}
\end{aligned} \tag{9}
$$

$$
\begin{aligned}
R_{XY}(t_1, t_2) &= E[X(t_1)Y(t_2)] \\
&= E[u(t_1 - v)\delta(t_2 - v)] \\
&= E[u(t_1 - t_2)] \\
&= \int_0^1 u(t_1 - t_2)\, dv \\
&= u(t_1 - t_2)
\end{aligned} \tag{10}
$$

7. Ensemble average:

$$
\begin{aligned}
E[X(t)] &= E[B\cos(wt) + A\sin(wt)] \\
&= E[B]\cos(wt) + E[A]\sin(wt) \\
&= 0
\end{aligned} \tag{11}
$$

Time average:

$$
\begin{aligned}
\langle X(t) \rangle_T &= \frac{1}{T}\int_0^T [B\cos(wt) + A\sin(wt)]dt \\
&= \frac{A - A\cos(Tw) + B\sin(Tw)}{Tw}
\end{aligned} \tag{12}
$$

$$
\begin{aligned}
E[\langle X(t) \rangle_T] &= E\left[\frac{A - A\cos(Tw) + B\sin(Tw)}{Tw}\right] \\
&= \frac{E[A] - E[A]\cos(Tw) + E[B]\sin(Tw)}{Tw} \\
&= 0
\end{aligned} \tag{13}
$$

Variance of the time average:

$$
\lim_{T \to \infty} \mathrm{Var}(\langle X(t) \rangle_T) = 0 \tag{14}
$$

$$
\mathrm{Var}(\langle X(t) \rangle_T) = E[\langle X(t) \rangle_T^2] - E[\langle X(t) \rangle_T]^2 \tag{15}
$$

Given that $E[\langle X(t) \rangle_T] = 0$, the variance simplifies to:

$$
\mathrm{Var}(\langle X(t) \rangle_T) = E[\langle X(t) \rangle_T^2] = \frac{-2(-1 + \cos(Tw))}{3Tw^2} \tag{16}
$$

The process $X(t)$ is ergodic in the mean for all non zero values of w.

8.

$$\begin{aligned} E[X_n] &= E\left[\sum_{i=1}^{n} Z_i\right] \\ &= \sum_{i=1}^{n} E[Z_i] \\ &= n \times (1 \times p + (-1) \times q) \\ &= n \times (p - q) \end{aligned} \tag{17}$$

$$Var(X_n) = E[X_n^2] - (E[X_n])^2 \tag{18}$$

Now, $E[X_n^2]$ is:

$$\begin{aligned} E[X_n^2] &= E\left[\left(\sum_{i=1}^{n} Z_i\right)^2\right] \\ &= E\left[\sum_{i=1}^{n} Z_i^2 + 2\sum_{i=1}^{n} \sum_{j=i+1}^{n} Z_i Z_j\right] \end{aligned} \tag{19}$$

Since Z_i are independent:

$$E[Z_i Z_j] = E[Z_i]E[Z_j] = (p - q)^2 \tag{20}$$

$$E[Z_i^2] = 1^2 \times p + (-1)^2 \times q = p + q \tag{21}$$

Using the above results:

$$E[X_n^2] = n(p + q) + 2n(n - 1)(p - q)^2 \tag{22}$$

Then:

$$Var(X_n) = n(p + q) + 2n(n - 1)(p - q)^2 - n^2(p - q)^2 \tag{23}$$

$$\begin{aligned} R_{XX}[m] &= E[X[n]X[n + m]] \\ &= E\left[\left(\sum_{i=1}^{n} Z_i\right)\left(\sum_{j=1}^{n+m} Z_j\right)\right] \\ &= n(p + q) + 2n(n - 1)(p - q)^2 + E\left[\left(\sum_{i=1}^{n} Z_i\right)\left(\sum_{j=n+1}^{n+m} Z_j\right)\right] \\ &= n(p + q) + 2n(n - 1)(p - q)^2 + nm(p - q)^2 \end{aligned} \tag{24}$$