Stochastic Processes

Week 06 (version 1.0)

Estimation Theory 01

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Outline of Week 06 Lectures

- Introduction to Estimation Theory
- Sufficient Statistic
- Minimal Sufficient Statistic
- Complete Sufficient Statistic
- Likelihood Principle
- Frequentist's Estimators: MLE, MM

Introduction to Estimation Theory

- **Estimation Theory:** is a branch of statistics that deals with estimating the values of parameters based on observed data that has a random component.
- In this course we focus on point estimation:

  Given $X = \{x_1, x_2, \ldots x_n\}$ where x_is are independent and identically distributed (i.i.d) observations with $f(x_i|\theta)$, we want to find an statistics $T(X) = \hat{\theta}$ that is a good estimator for θ.

Introduction to Estimation Theory

- Three basic Questions:
  1) Do we need all the i.i.d observations to estimate θ?
  2) What do we mean by “good estimator”?
  3) Do we need prior information on θ (i.e. *f*(θ)) to estimate it?
- Answers:
  1) Not necessarily! We may use Sufficient Statistic (SS); a function or statistic of observations, instead.
  2) The goodness of an estimator is measured by three properties: unbiasedness, efficiency (minimum variance) and consistency.

Introduction to Estimation Theory

- Unbiasedness:

An estimator $\hat{\theta}$ is said to be unbiased if its expected value is identical to θ; E ($\hat{\theta}$) = θ.

- Efficiency:

If two competing estimators are both unbiased, the one with the smaller variance is said to be relatively more efficient.

- Consistency:

If an estimator $\hat{\theta}$ approaches the parameter θ closer and closer as the sample size *n* increases, $\hat{\theta}$ is said to be a consistent estimator of θ (not a rigorous definition).

Introduction to Estimation Theory

3) The frequentist believe we do not need prior information on θ (i.e. $f(\theta)$) to estimate it. However, the Bayesian believe we do need prior information on θ.

In the following we focus on Sufficient Statistic.

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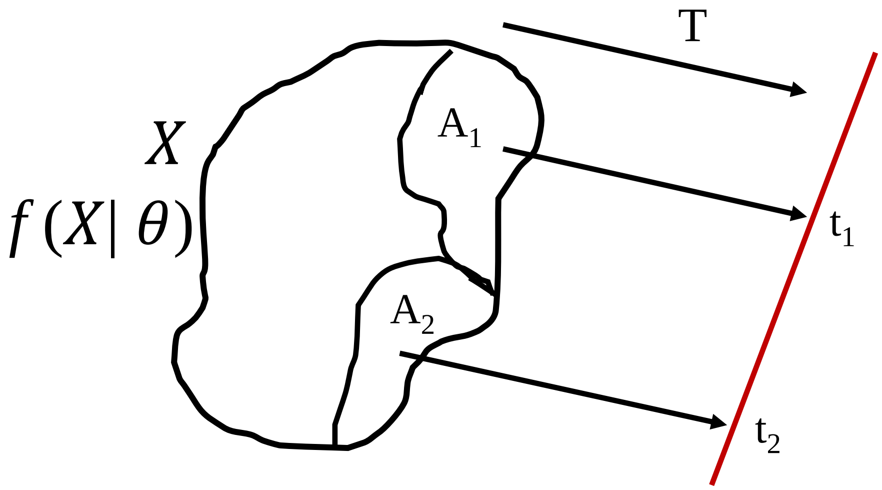
Sufficient Statistic (SS)

Assume the statistic T partitions the sample space into sets.



$A_i = \{x : T(x) = t_i\}$

Goal of SS: Data reduction without discarding information about θ. Examples of statistics:

$$T(X) = 2$$
$$T(X) = X$$

Sufficient Statistic

- A statistic T(X) is a sufficient statistic for θ if the conditional density of X given the value of T(X) does not depend on θ.
- In other words, if T(X) is a sufficient statistic for θ then any inference about θ should depend on the sample X only through T(X); meaning $\hat{\theta}$ is a function of T(X).
- How to find sufficient statistics for θ?

Sufficient Statistic

Factorization Theorem:

Let $f(x|\theta)$ be the pdf of X.

$T(X)$ is a sufficient stat for θ iff $\exists$ functions g and h such that:

$$f(x|\theta) = g(T(x)|\theta)\, h(x) \qquad \forall x \in \chi, \qquad \theta \in \Theta$$

proof: (discrete case)

$\Rightarrow$: Assume T is a sufficient statistic:

$$f(x|\theta) = P_\theta\big(X = x, T(X) = T(x)\big)$$

$$= \underbrace{P_\theta\big(T(X) = T(x)\big)}_{g(T(x)|\theta)} \underbrace{P_\theta(X = x|T(X) = T(x))}_{h(x)}$$

Sufficient Statistic

$\Leftarrow$: Assume factorization holds, let $q(t|\theta)$ be the pmf of T(X)

Let $A_t = \{y: T(y) = t\}$

$$q(t|\theta) = P_\theta(T(X) = t) = \sum_{x \in A_t} f(x|\theta) = \sum_{x \in A_t} g(T(x)|\theta)h(x)$$

$$P_\theta(X = x|T(X) = T(x)) = \frac{P_\theta\big(X=x, T(X)=T(x)\big)}{P_\theta\big(T(X)=T(x)\big)} = \frac{P_\theta(X=x)}{q(t|\theta)}$$

$$= \frac{g\big(T(x)\big|\theta\big)h(x)}{g\big(T(x)\big|\theta\big)\sum_{x\in A_t} h(x)} = \frac{h(x)}{\sum_{x\in A_t} h(x)}$$ does not depend on θ.

Sufficient Statistic

Example: $x_1, \ldots, x_n$ be i.i.d Bernouli(θ), $\quad 0 < \theta < 1$.

Then $T(x) = \sum_{i=1}^{n} x_i$ is a sufficient statistic for θ.

$$f(x|\theta) = \prod_{i=1}^{n} \theta^{x_i}(1-\theta)^{1-x_i} = \theta^{\sum x_i}(1-\theta)^{n-\sum x_i}$$

$g(t|\theta) \coloneqq \theta^t(1-\theta)^{n-t}$

$h(x) \coloneqq 1$

Sufficient Statistic

Example: $x_1, \ldots, x_n$ be i.i.d U(0, θ).

$$f(x_1, \ldots, x_n|\theta) = \begin{cases} \frac{1}{\theta^n} & all\ x_i\ in\ [0, \theta] \\ 0 & o.w. \end{cases}$$

Recall: $I_A(x) = \begin{cases} 1 & if\ x \in A \\ 0 & o.w. \end{cases}$

Let: $T(x) = \max_i x_i$

Define: $g(t|\theta) \coloneqq \frac{1}{\theta^n} I_{(-\infty,\theta]}(t)$ $\qquad h(x) = I_{[0,+\infty)}\left(\min_i x_i\right)$

$$\Rightarrow g(T(x)|\theta)h(x) = \frac{1}{\theta^n} I_{(-\infty,\theta]}\left(\max_i x_i\right) \cdot I_{[0,+\infty)}\left(\min_i x_i\right) = f(x_1, \ldots, x\_n|\theta)$$

$\Rightarrow T(X)$ is sufficient statistic.

Sufficient Statistic

Example: $x_1, \dots, x_n$ be i.i.d Normal(μ, δ^2).

$$f(x|\mu,\delta^2) = (2\pi\delta^2)^{-\frac{n}{2}} \exp\left(-\frac{\sum_{i=1}^{n}(x_i - \bar{x})^2 + n(\bar{x} - \mu)^2}{2\delta^2}\right)$$

We show that following t_1 and t_2 together is a sufficient statistic.

$$t_1 = \sum_{i=1}^{n}(x_i - \bar{x})^2, \qquad t_2 = \bar{x}$$

need: $g(t_1, t_2|\theta)$

$$g(t|\theta) = g(t_1, t_2|\mu, \delta^2) = (2\pi\delta^2)^{-\frac{n}{2}} \exp\left(-\frac{\left(t_2 + n(t_1 - \mu)\right)}{2\delta^2}\right)$$

$h(x) = 1$

$\Rightarrow T(X)$ is sufficient statistic.

Sufficient Statistic

Exponential Family:

Family of pdfs or pmfs is called a k-parameter exponential family if:

$$f(x|\theta) = h(x)\, c(\theta) \exp\left(\sum_{i=1}^{k} w_i(\theta)\, t_i(x) \right)$$

Example: $x_1, \ldots, x_n$ be i.i.d Bernouli(θ), $0 < \theta < 1$.

$$f(x|\theta) = \theta^{\sum x_i} (1-\theta)^{n - \sum x_i} = \exp\left(\ln\theta \sum_{i=1}^{n} x_i + \ln(1-\theta)\left(n - \sum_{i=1}^{n} x_i \right) \right)$$

$$= \exp\left(\ln\frac{\theta}{1-\theta} \sum_{i=1}^{n} x_i + n\ln(1-\theta) \right) = \exp(n\ln(1-\theta)) \cdot \exp\left(\ln\frac{\theta}{1-\theta} \sum_{i=1}^{n} x_i \right)$$

$k = 1,$ $\quad h(x) = 1\,,$ $\quad c(\theta) = \exp(n\ln(1-\theta)),$ $\quad t_1 = \sum_{i=1}^{n} x_i\,,$ $\quad w_1(\theta) = \ln\frac{\theta}{1-\theta}$

Sufficient Statistic

Example: $x_1, \dots, x_n$ be i.i.d Normal(μ, δ^2).

$$f(x|\mu,\delta^2) = \frac{1}{\sqrt{2\pi\delta^2}} \exp\left(-\frac{(x-\bar{\mu})^2}{2\delta^2}\right) = \frac{1}{\sqrt{2\pi}} \frac{1}{\delta} \exp\left(-\frac{\mu^2}{2\delta^2}\right) \exp\left(-\frac{x^2}{2\delta^2} + \frac{\mu x}{\delta^2}\right)$$

Exponential family:

$$f(x|\theta) = h(x)\, c(\theta) \exp\left(\sum_{i=1}^{k} w_i(\theta)\, t_i(x)\right)$$

$\Rightarrow$

$$k = 2, \qquad h(x) = 1\,, \qquad c(\mu,\delta^2) = \frac{1}{\sqrt{2\pi}} \frac{1}{\delta} \exp\left(-\frac{\mu^2}{2\delta^2}\right),$$

$$t_1(x) = \frac{x^2}{2}, \qquad w_1(\mu,\delta^2) = \frac{1}{\delta^2}$$

$$t_2(x) = x, \qquad w_2(\mu,\delta^2) = \frac{\mu}{\delta^2}$$

Sufficient Statistic

Sufficient statistic for exponential family:

Let $x_1, \dots, x_n$ be i.i.d observations from a pdf or pmf $f(x|\theta)$. Suppose $f(x|\theta)$ belongs to the exponential family:

$$f(x|\theta) = h(x)\, c(\theta) \exp\left(\sum_{i=1}^{k} w_i(\theta)\, t_i(x)\right)$$

Then

$T(X) = (\sum_{i=1}^{n} t_1(x_i), \sum_{i=1}^{n} t_2(x_i), \dots, \sum_{i=1}^{n} t_k(x_i))$ is a sufficient statistic for θ.

Example: $x_1, \dots, x_n$ be i.i.d Normal(μ, δ^2).

$t_1(x) = -\dfrac{x^2}{2}$ $\qquad$ $t_2(x) = x$

Sufficient Statistic

$\Rightarrow T(X) = \left(-\frac{1}{2}\sum_{i=1}^{n} x_i^2, \sum_{i=1}^{n} x_i\right)$ is sufficient statistic for (μ, δ^2)

$T'(X) = (\sum_{i=1}^{n}(x_i - \bar{x})^2, \bar{x})$

$T(X) = T(Y) \quad iff \quad T'(X) = T'(Y)$

Results:

1) T(X) = X is a sufficient statistic.

Proof:

$f(x|\theta) = f(T(x)|\theta)h(x)$

$T(x) = x, \qquad h(x) = 1$

Sufficient Statistic

2) Any one-to-one function of a sufficient statistic is also a sufficient statistic.

Proof: Suppose T is a sufficient statistic.

Define $T^*(x) = r(T(x))$ where r is one-to-one and has inverse r^{-1}

$$f(x|\theta) = g(T(x)|\theta)h(x) = g\left(r^{-1}\left(T^*(x)\right)|\theta\right)h(x)$$

Define $g^*(t|\theta) = g(r^{-1}(t)|\theta)h(x)$

$$\Rightarrow f(x|\theta) = g^*(T^*(x)|\theta)\ h(x)$$ so T^* is a sufficient static for θ.

Example: $x_1, \dots, x_n$ be i.i.d Bernouli(θ), $\quad 0 < \theta < 1$.

All of the following are sufficient statics for θ:

$$T_1(X) = \sum_{i=1}^{n} x_i\,, \qquad T_2(X) = \left(x_{(1)}, x_{(2)}, \dots, x_{(n)}\right), \qquad T_3(X) = (x_1, x_2, \dots, x_n)$$

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Minimal Sufficient Statistic

Minimal sufficient statistic:

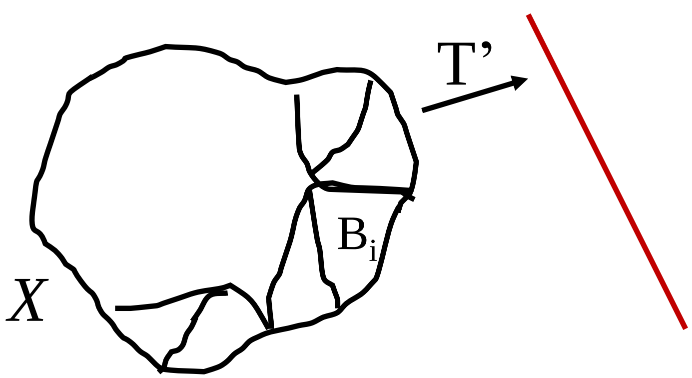
A sufficient statistic $T(X)$ is called minimal sufficient statistic, if for any other sufficient statistic $T'(X)$, $T(X)$ is a function of $T'(X)$.

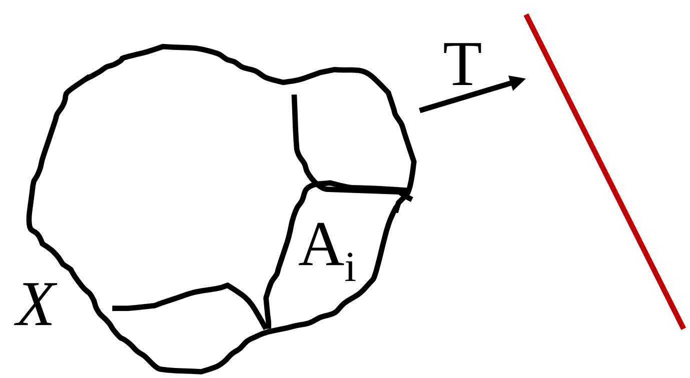
It achieve maximum possible data reduction without losing info about θ.

T partitions χ into sets; $A_t = \{\underline{X} : T(\underline{X}) = t\}$

T'partitions χ into sets; $B_{t'} = \{\underline{X} : T'(\underline{X}) = t'\}$

Each set $B_{t'} \subset$ some set A_t





Minimal Sufficient Statistic

Theorem:

Let $f(x|\theta)$ be pdf or pmf. Suppose that for any 2 sample points $\underline{X}$ and $\underline{Y}$ the ratio:

$$\frac{f(\underline{X}|\theta)}{f(\underline{Y}|\theta)}$$

is constant as a function of θ iff $T(X) = T(Y)$:

$\boldsymbol{T(X)}$ **is a minimal sufficient statistic for** $\boldsymbol{\theta}$, iff the above holds.

Proof: assume $f(x|\theta) > 0$

Let $I = \{t : t = T(x) \; for \; some \; x \in \chi\}$

$A_t = \{\underline{X} : T(\underline{X}) = t\}$

Minimal Sufficient Statistic

for each A_t, choose a fix element $X_t \in A_t$. For any $\underline{X}$, let $X_{T(x)}$ be the fixed element that is in the same A_t as $\underline{X}$, Hence:

$$T\left(\underline{X}\right) = T(X_{T(X)})$$

$$\Rightarrow \frac{f(\underline{X}|\theta)}{f(X_{T(x)}|\theta)}$$ is constant as a function of θ.

$$g(t|\theta) \coloneqq f\left(X_{T(x)}\middle|\theta\right)$$

$$f(x|\theta) = \frac{f\left(X_{T(x)}\middle|\theta\right)\ f(\underline{x}|\theta)}{f\left(X_{T(x)}\middle|\theta\right)} = g(T(x)|\theta)\ h(x)$$

$\Rightarrow T(x)$ is sufficient.

Minimal Sufficient Statistic

$\Leftarrow$ Let T' be an arbitrary sufficient statistic. Then from factorization theorem:

$\exists$ functions $g', h' \quad s.t. \quad f(x|\theta) = g'(T'(x)|\theta)\, h'(x)$

For any 2 sample points like $\underline{x}, \underline{y}$ with $T'(x) = T'(y)$:

$$\frac{f(x|\theta)}{f(y|\theta)} = \frac{g'\left(T'(x)\middle|\theta\right) h'(x)}{g'\left(T'(y)\middle|\theta\right) h'(y)} = \frac{h'(x)}{h'(y)}$$ which is a constant as a function of θ.

So by the assumption about $T(x)$ we have: $T(\underline{x}) = T(\underline{y})$.

Therefore, T is a function of T'.

Hence T is minimal.

Minimal Sufficient Statistic

Example: $x_1, \dots, x_n$ be i.i.d Bernoulli(θ), $0 < \theta < 1$

$$f(x|\theta) = \prod_{i=1}^{n} \theta^{x_i}(1-\theta)^{1-x_i} = \theta^{\sum x_i}(1-\theta)^{n-\sum x_i}$$

$$\Rightarrow \frac{f(x|\theta)}{f(y|\theta)} = \theta^{\sum x_i - \sum y_i}(1-\theta)^{\sum y_i - \sum x_i}$$

need: $\sum x_i - \sum y_i = 0$

So $T(X) = \sum_{i=1}^{n} x_i$ is minimal sufficient for θ.

Minimal Sufficient Statistic

Example: $x_1, \dots, x_n$ be i.i.d Normal(μ, δ^2).

$$f(x|\mu, \delta^2) = (2\pi\delta^2)^{-\frac{n}{2}} \exp\left(-\frac{\sum_{i=1}^{n}(x_i - \bar{x})^2 + n(\bar{x} - \mu)^2}{2\delta^2}\right)$$

$$\frac{f(x|\mu, \delta^2)}{f(y|\mu, \delta^2)} = \exp\left(\frac{-n(\bar{x}^2 - \bar{y}^2) + 2n\mu(\bar{x} - \bar{y}) - (n-1)(\sum_{i=1}^{n}(x_i - \bar{x})^2 - \sum_{i=1}^{n}(y_i - \bar{y})^2)}{2\delta^2}\right)$$

Need:

$$\bar{x} = \bar{y}$$

$$\sum_{i=1}^{n}(x_i - \bar{x})^2 = \sum_{i=1}^{n}(y_i - \bar{y})^2$$

So $(\bar{x}, \sum_{i=1}^{n}(x_i - \bar{x})^2)$ is a minimal sufficient statistic for θ.

But it is not unique. E.g. $(\bar{x}, s^2)$ is also a minimal sufficient statistic for θ.

Minimal Sufficient Statistic

Any 1-1 function of a minimal sufficient statistic is a minimal sufficient statistic.

Example: $x_1, \dots, x_n$ be i.i.d $U(\theta, \theta+1)$

$$f(x|\theta) = \begin{cases} 1 & all\ x_i\ in\ (\theta, \theta+1) \\ 0 & o.w. \end{cases} = \begin{cases} 1 & \max(x_i) - 1 < \theta < \min(x_i) \\ 0 & o.w. \end{cases}$$

$\frac{f(x|\theta)}{f(y|\theta)}$ is constant as a function of θ iff $\begin{cases} \max(x_i) = \max(y_i) \\ \min(x_i) = \min(y_i) \end{cases}$

Hence, $T(X) = \left(x_{(1)}, x_{(n)}\right)$ is a minimal sufficient statistic for θ.

Note: $T'(x) = \left(x_{(n)} - x_{(1)}, \frac{x_{(1)} + x_{(n)}}{2}\right)$ is also minimal sufficient statistic.

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Complete Sufficient Statistic

Def: let $f(t|\theta)$ be family of pdfs (pmfs) for a statistic $\mathrm{T}(x)$, the family of probability distributions is called complete if:

$$E_\theta \, g(T) = 0 \quad \forall\theta$$

$$\Rightarrow p_\theta(g(T) = 0) = 1 \quad \forall\theta$$

or $\mathrm{T}(x)$is a complete statistic.

Note: completeness is a property of the family of distributions not a particular distribution.

Complete Sufficient Statistic

Example: Let X be a random sample of size n such that each X_i has the same Bernoulli distribution with parameter p. Let T be the number of 1s observed in the sample, i.e.

$$T = \sum_{i=1}^{n} X_i$$

T is a statistic of X which has a binomial distribution with parameters (n, p). If the parameter space for p is (0,1), then T is a complete statistic:

$$\mathrm{E}_p(g(T)) = \sum_{t=0}^{n} g(t)\binom{n}{t} p^t (1-p)^{n-t} = (1-p)^n \sum_{t=0}^{n} g(t)\binom{n}{t}\left(\frac{p}{1-p}\right)^t$$

neither p nor $1 - p$ can be 0.

Complete Sufficient Statistic

Hence: $E_p(g(T)) = 0$ iff:

$$\sum_{t=0}^{n} g(t) \binom{n}{t} \left(\frac{p}{1-p} \right)^t = 0$$

Replacing $p/(1-p)$ by r:

$$\sum_{t=0}^{n} g(t) \binom{n}{t} r^t = 0$$

The range of r is the positive reals. Also, E(g(T)) is a polynomial in r and, therefore, can only be identical to 0 if all coefficients are 0, that is, $g(t) = 0$ for all t.

Complete Sufficient Statistic

- It is important to notice that the result that all coefficients must be 0 was obtained because of the range of *r*.
- For example, for a single observation and a single parameter value; if $n = 1$ and the parameter space is $\{0.5\}$, T is not complete: $g(t) = 2\,(t - 0.5)$ and then, $E(g(T)) = 0$ although $g(t)$ is not 0 for $t = 0$ nor for $t = 1$.

Theorem: (exponential family)

Let $x_1, \dots, x_n \quad iid \quad F(x|\theta) \qquad\qquad f(x|\theta) = h(x)\, c(\theta) \exp(\sum w_i(\theta) t_i(x))$

Suppose that the range of $(w_1(\theta), \dots, w_k(\theta))$ contains an n dimensional rectangle.

Then: $T(\underline{x}) = \left(\sum_{j=1}^{n} t_1(x_j), \dots, \sum_{j=1}^{n} t_k(x_j)\right)$ is complete.

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The Likelihood Principle

The likelihood principle:

Def: $\underline{X} \sim f(x|\theta)$

Then given $\underline{X} = \underline{x}$ observed, the function of θ defined by:

$$L\left(\theta \middle| \underline{X}\right) = f(\underline{X}|\theta)$$

Is called the likelihood function.

Interpretation:

1) X discrete

$L(\theta|X) = p_\theta(\underline{X} = \underline{x})$

$L_1(\theta_2|\underline{X}) > L_2(\theta_2|\underline{X})$

Sample had a higher likelihood of occurring if $\theta = \theta_1$ then $\theta = \theta_2$.

The Likelihood Principle

2) X continuous (real valued pdf)

for small ε:

$$2\varepsilon L(\theta|X) = 2\varepsilon f(X|\theta) \cong p_\theta(X - \varepsilon < X < X + \varepsilon)$$

$$\frac{L(\theta_1|X)}{L(\theta_0|X)} = \frac{p_{\theta_1}(X - \varepsilon < X < X + \varepsilon)}{p_{\theta_0}(X - \varepsilon < X < X + \varepsilon)} > 1 \ \ ?$$

approx. the same interpretation as discrete.

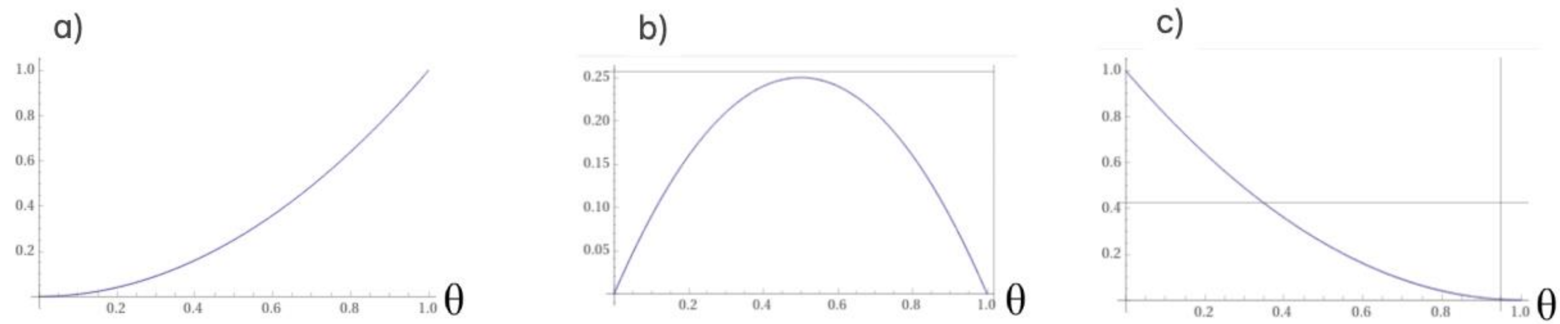
Example: $x_1, \ldots, x_n \quad iid \;\; Bernoulli(\theta)$

$$L(\theta \mid x) = f(x \mid \theta) = \theta^{\Sigma x_i}(1-\theta)^{n-\Sigma x_i}$$

Let $n = 2$

The Likelihood Principle

(a) $\Sigma x_i = 2 \Rightarrow L(\theta \mid x) = \theta^2$
(b) $\Sigma x_i = 1 \Rightarrow L(\theta \mid x) = \theta(1-\theta)$
(c) $\Sigma x_i = 0 \Rightarrow L(\theta \mid x) = (1-\theta)^2$



consider $L\left(\frac{3}{4} \mid x\right) / L\left(\frac{1}{4} \mid x\right)$

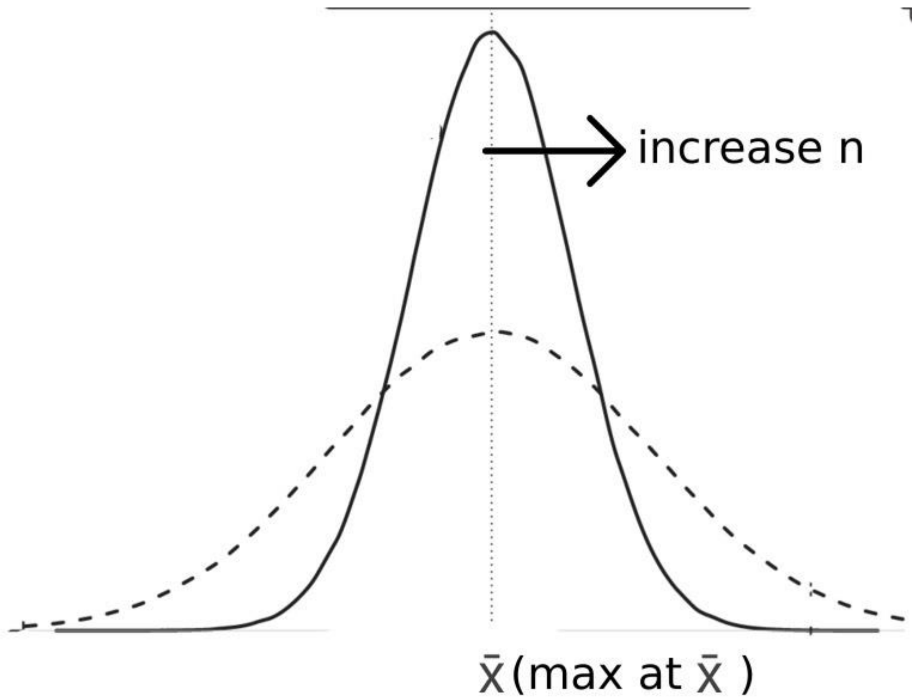
(a) $$\frac{L(3/4|x)}{L(1/4|x)} = \begin{cases} 9 & \text{when } \sum x_i = 2 \\ 1 & \text{whan } \sum x_i = 1 \\ \frac{1}{9} & \text{when } \sum x_i = 0 \end{cases}$$

The Likelihood Principle

Example: $x_1, \ldots, x_n$ iid $N(\mu,\ \delta^2)$ **.** Assume δ^2 is fixed.

$$L(\mu \mid x) = f(x \mid \mu) = \left(2\pi\delta^2\right)^{-n/2} e^{-\frac{1}{2\delta^2}\left[\sum_{-} (x_i - \bar{x})^2 + n(\bar{x}-\mu)^2\right]}$$

$$= k(x) e^{-n(\bar{x}-\mu)^2/2\delta^2}$$



The Likelihood Principle

Likelihood principle:

If X and Y are two sample points (with the same parameter θ) s.t. $L(\theta|X)$ is proportional to $L(\theta|Y)$:

$$L(\theta|X) = C(X,Y)\,L(\theta|Y) \quad \forall\theta$$

Then the conclusions drown from X and Y about θ should be identical.

Idea: use the likelihood function to compare the “probability” of various parameter values.

if $L(\theta_2|X) = 2L(\theta_1|X)$ θ_2 is twice as likely as θ_1 and:

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Frequentist's Estimators

Def: A point estimator is any statistic $T(x)$.

Estimator: function of samples (function of SS).

Estimate: actual value of the estimator.

Methods of finding estimators for this course:

(1) Maximum Likelihood Estimator (MLE) ~ (frequentist)

(2) Method of Moments (MM) ~ (frequentist)

(3) UMVUE ~ (frequentist)

(4) Maximum APosteriori (MAP) ~ (Bayes)

(5) Bayes Minimum Risk ~ (Bayes)

Maximum Likelihood Estimator: MLE

Maximum likelihood estimator (MLE):

$$L(\theta|X) = L(\theta_1, \dots, \theta_k | X_1, \dots, X_n) = \prod_{i=1}^{n} f(X_i|\theta)$$

Def:

for each $\underline{X}$, let $\hat{\theta}(X)$ be the value which maximizes $L(\theta|X)$

then, $\hat{\theta}(X)$ is the maximum likelihood estimator (MLE) of θ.

$$\hat{\theta}_{ML}(X) = Arg\ Max\ (L(\theta|X))$$

Log likelihood:

use $\log L(\theta|X)$.

Maximum Likelihood Estimator: MLE

How to find MLE's:

(1) Differentiation

if $L(\theta|X)$ is differentiable in θ_i, possible θ_i's are solutions to:

$$\frac{\partial}{\partial\theta_i}L(\theta|X) = 0 \quad , \; \mathrm{i} = 1, \ldots, \mathrm{k}$$

a) 1-dimension

solve $\frac{\partial}{\partial\theta}L(\theta|X) = 0$ for $\hat{\theta}$

check $\frac{\partial^2}{\partial\theta^2}L(\theta|X) < 0$ for $\theta = \hat{\theta}$

(check boundaries)

Maximum Likelihood Estimator: MLE

Example: $x_1, \dots, x_n \quad iid \quad Bern(\theta)$

$$\mathrm{L}(\theta \mid x) = \theta^{\sum x_i}(1-\theta)^{n-\sum x_i}$$

$$\log L(\theta \mid x) = \sum x_i \log\theta + (n - \sum x_i)\log(1-\theta)$$

$$\frac{\partial \log L(\theta \mid x)}{\partial\theta} = \frac{\sum x_i}{\theta} - \frac{n-\sum x_i}{1-\theta} = 0 \Rightarrow \hat{\theta} = \bar{x}$$

$$\frac{\partial^2 \log L}{\partial\theta^2} = -\frac{\sum x_i}{\theta^2} - \frac{n-\sum x_i}{(1-\theta)^2} < 0 \; @\theta = \hat{\theta}$$

check bounderies; $\sum x_i = 0, \sum x_i = n$

$$\log L(\theta \mid x) = \begin{cases} n\log(1-\theta) & if \sum x_i = 0 \\ n\log(\theta) & if \sum x_i = \mathrm{n} \end{cases}$$

Maximum Likelihood Estimator: MLE

b) 2-dimensions

solve $\frac{\partial}{\partial \theta_1} L(\theta_1, \theta_2 | X) = 0$

, $\frac{\partial}{\partial \theta_2} L(\theta_1, \theta_2 | X) = 0 \quad for \;\; \theta_1, \theta_2$

check that $\frac{\partial^2}{\partial {\theta_1}^2} L(\theta_1, \theta_2 | X) < 0$ for $\theta_1 = \hat{\theta}_1,\ \theta_2 = \hat{\theta}_2$

or: $\frac{\partial^2}{\partial {\theta_2}^2} L(\theta_1, \theta_2 | X) < 0$ for $\theta_1 = \hat{\theta}_1,\ \theta_2 = \hat{\theta}_2$

and: $\frac{\partial^2}{\partial {\theta_1}^2} L(\theta_1, \theta_2 | X) \ \frac{\partial^2}{\partial {\theta_2}^2} L(\theta_1, \theta_2 | X) - \left[\frac{\partial^2}{\partial \theta_1 \partial \theta_2} L(\theta_1, \theta_2 | X)\right]^2 > 0$

for $\theta_1 = \hat{\theta}_1,\ \theta_2 = \hat{\theta}_2$.

Maximum Likelihood Estimator: MLE

Example: $x_1, \dots, x_n \quad iid \quad N(\mu,\ \delta^2)$

$$\log L(\mu, \delta^2 \mid x) = -\frac{n}{2}\log 2\pi - \frac{n}{2}\log s^2 - \frac{1}{2\delta^2}\sum(x_i - \mu)^2$$

$$\frac{\partial}{\partial\mu}\log L = \frac{1}{\delta^2}\sum(x_i - \mu) = 0 \Rightarrow \hat{\mu} = \bar{x}$$

$$\frac{\partial}{\partial\delta^2}\log L = -\frac{n}{2\delta^3} + \frac{1}{2\delta^4}\sum(x_i - \mu)^2 = 0 \Rightarrow \hat{\delta}^2 = \sum(x_i - \bar{x})^2$$

(i) $$\frac{\partial^2}{\partial\mu^2}\log L = -\frac{n}{\delta^2}$$

(ii) $$\frac{\partial^2}{\partial(s^2)^2}\log L = \frac{n}{2\delta^4} - \frac{1}{\delta^6}\sum(x_i - \mu)^2$$

(ii) $$\frac{\partial^2}{\delta\mu\partial\delta^2}\log L = -\frac{1}{\delta^4}\sum(x_i - \mu)$$

Maximum Likelihood Estimator: MLE

$$\frac{1}{\delta^6}\left[-\frac{n^2}{2}+\frac{n}{\delta^2}\sum(x_i-\mu)^2-\frac{1}{\delta^2}\left(\sum(x_i-\mu)\right)^2\right]\Big|_{\delta^2=\hat{\delta}^2}^{\mu=\hat{\mu}_i}$$

$$=\frac{1}{\hat{\delta}^6}\left[-\frac{n^2}{2}+\frac{n}{\hat{\delta}^2}n\hat{\delta}^2-\frac{1}{\hat{z}^2}(0)\right]=\frac{n^2}{2\hat{s}^2}>0$$

Maximum Likelihood Estimator: MLE

How to find MLE's:

(2) Direct maximization

- find global upper bound on likelihood function
- show bound is attained

Example: $x_1, \ldots, x_n \quad iid \quad N(\mu, 1)$

$$L(\mu \mid x) = \left(\frac{1}{\sqrt{2\pi}}\right)^2 e^{-\frac{1}{2}\sum (x_i - \mu)^2}$$

Recall for any number a: $\sum (x_i - \bar{x})^2 \leqslant \sum (x_i - a)^2$

$$\Rightarrow L(\mu \mid \underline{x}) \leqslant L(\bar{x} \mid \underline{x}) \Rightarrow \hat{\mu} = \bar{x}$$

Maximum Likelihood Estimator: MLE

(3) Numerically (by computer)

With or without (1) and (2)

Example: $x_1, \ldots, x_n$ $\quad iid \quad truncated\ poisson$:

$$p[x_i = r] = \frac{e^{-m} m^r}{(1 - e^{-m}) r!}, m \leq 0,1, \ldots$$

$$L(m \mid x) = \Pi_{i=1}^{n} \frac{e^{-m} m^{x_i}}{(1 - e^{-m}) x_i!} = \left(\frac{e^{-m}}{i - e^{-m}} \right)^r m^{\sum x_i} \Pi_{i=1}^{n} \frac{1}{x_i!}$$

$$\log L = -mn - n\log(1 - e^{-m}) + \sum x_i \lg m - \sum \log(x_i!)$$

$$\frac{\partial \log L}{\partial m} s + n - \frac{ne^{-m}}{1 - e^{-m}} + \frac{\Sigma x_i}{m} = 0 \Rightarrow \hat{m} = ?$$

Define: $\phi(m) = \frac{\partial \log L}{\partial m}$, need $\hat{m}$ s/t $\phi(\hat{m}) = 0$

Maximum Likelihood Estimator: MLE

Let m_0 be an initial estimate for $\hat{m}$.

$$0 \approx \phi(\hat{m}) \approx \phi(m_0) + (\hat{m} - m_0)\,\phi'(m_0)$$

$$\hat{m} \approx m_0 - \frac{\phi(m_0)}{\phi'(m_0)}$$

(1) Choose an initial estimate m_0

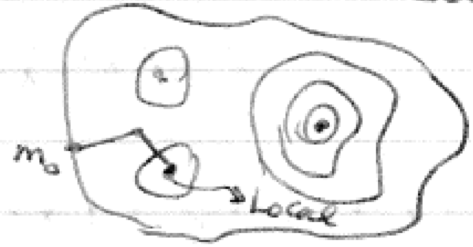
(2) Define a sequence $\{m_k\}$ of estimates by:

$$m_{k+1} = m_k - \frac{\phi(m_k)}{\phi'(m_k)}\,, k = 0,1,2, \ldots$$

(3) Stop when $|m_{k+1} - m_k| < \varepsilon$

Let $\hat{m} = m_k$

• problem;
Local instead of global.



Maximum Likelihood Estimator: MLE

Note: maximization takes place only over the range of parameter values.

Example: $x_1, \ldots, x_n \quad iid \quad N(\mu, 1) \quad but \quad \mu \geq 0$

$$\hat{\mu} = \bar{x} \text{ what if } \bar{x} < 0?$$

$$\hat{\mu} = 0 \text{ if } \bar{x} < 0 \;\Rightarrow \hat{\mu} = \begin{cases} \bar{x}, & \bar{x} \geqslant 0 \\ 0 & , \bar{x} < 0 \end{cases}$$

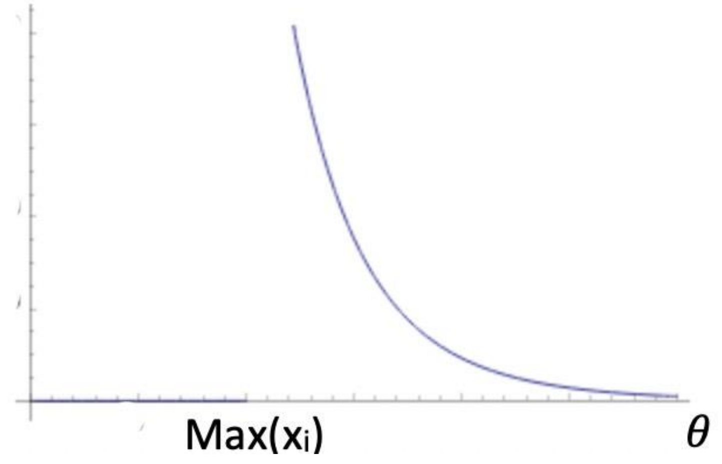
Maximum Likelihood Estimator: MLE

Note: maximization can occur on boundaries.

Example: $x_1, \ldots, x_n \quad iid \quad U(0, \theta)$

$$L(\theta \mid \mathrm{X}) = \begin{cases} \dfrac{1}{\theta^n} & \text{if } \theta \geqslant \max(x_i) \\ 0 & \text{else} \end{cases}$$

$$\therefore \hat{\theta}_{m\mathrm{L}E} = \max(x_i)$$



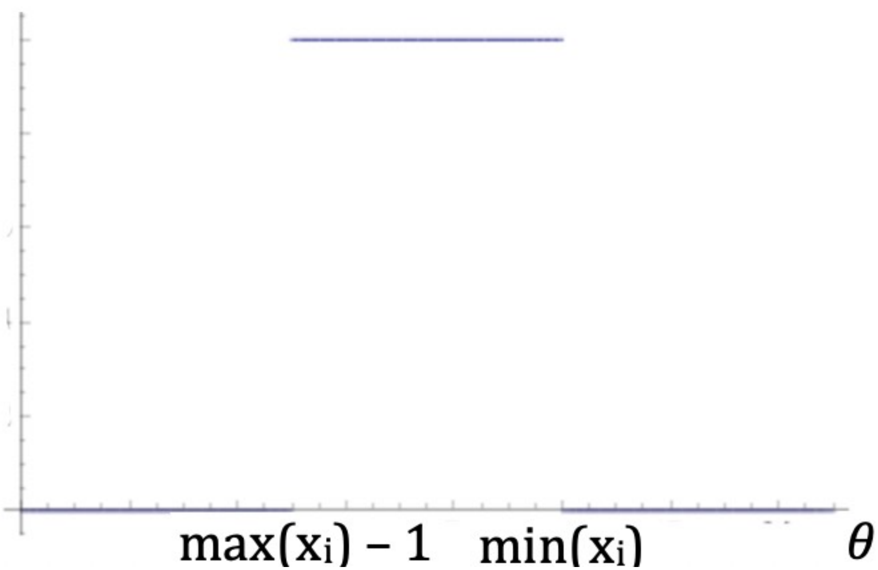
Note: maximum likelihood estimate may not be unique.

Maximum Likelihood Estimator: MLE

Note: maximum likelihood estimate may not be unique.

Example: $x_1, \ldots, x_n \quad iid \quad U(\theta, \theta + 1)$

$$L(\theta \mid \underline{x}) = \begin{cases} 1 & \max x_i - 1 < \theta < \min x_i \\ 0 & 0, \omega. \end{cases}$$



$\therefore \hat{\theta}$ = any value in the interval $\left(\max(x_i) - 1, \min(x_i)\right)$

Maximum Likelihood Estimator: MLE

Note: MLE's can be numerically unstable.

Example: $x_1, \ldots, x_n \quad iid \quad Bin(k,p) \;;\; k,p\; unknowns$

Can show:

$$\text{if } \underline{x} = (16,18,22,25,27) \Rightarrow \hat{k} = 99$$
$$\text{if } \underline{x} = (16,18,22,25,28) \Longrightarrow \hat{k} = 190$$

Maximum Likelihood Estimator: MLE

Theorem: (invariance property)

If $\hat{\theta}$ is the MLE of θ, then for any function $r(\theta)$, $r(\hat{\theta})$ is the MLE of $r(\theta)$.

Example: $x_1, \dots, x_n \quad iid \quad N(\mu, 1)$

$\overline{X}$ is the MLE of μ , then $\overline{X}^2$ is the MLE of μ^2.

Method of Moments

Method of moments:

$x_1, \dots, x_n \quad iid \;\; f(x|\theta_1, \dots, \theta_k)$

Equate the first k sample moments to the k first population moments.

Let $\quad m_1 = \frac{1}{n}\sum X_i \qquad\qquad \mu_1 = E(X)$

$m_2 = \frac{1}{n}\sum X_i^2 \qquad\qquad \mu_2 = E(X^2)$

$\vdots \qquad\qquad\qquad \vdots$

$m_k = \frac{1}{n}\sum X_i^k \qquad\qquad \mu_k = E(X^k)$

$m_j = \mu_j(\theta_1, \dots, \theta_k)$

Let $\quad m_1 = \mu_1(\theta_1, \dots, \theta_k)$

$\vdots$

$m_k = \mu_k(\theta_1, \dots, \theta_k)$ solve for $\theta_1, \dots, \theta_k$

Method of moments

Example: $x_1, \dots, x_n \quad iid \quad N(\mu,\ \delta^2)$

$$m_1 = \frac{1}{n}\sum x_i \qquad\qquad \mu_1 = \mu$$

$$m_2 = \frac{1}{n}\sum x_i^2 \qquad\qquad \hat{\mu}_2 = \delta^2 + \mu^2$$

$$\bar{x} = \mu, \frac{1}{n}\sum x_i^2 = s^2 + \mu^2 \Rightarrow \hat{\mu} = \bar{x} + \hat{\delta}^2 = \frac{1}{n}\Sigma_{i=1}^{n}\left(x_i - \bar{x}\right)^2$$

Example: $x_1, \dots, x_n \quad iid \quad binomial(k, p)$ both unknown

$$\bar{x} = kp$$

$$\frac{1}{n}\sum x_i^2 = kp(1-p) + k^2p^2$$

Solving to get: $\hat{k} = \dfrac{\bar{x}^2}{\left[\bar{x} - \frac{1}{n}\sum (x_i - \bar{x})^2\right]}$

$$\hat{p} = \frac{\bar{x}}{\hat{k}}$$

Method of moments

Note: this method can also be used for moment matching.

-match moments of distributions of statistics to obtain approximation to distributions.

Example: $x_1, \ldots, x_n \quad iid \quad p(\lambda)$

$$m_1 = \frac{1}{n}\Sigma x_i$$

(1) $E(x_1) = \lambda$

(2) $E(x_1^2) = \lambda + \lambda^2$

$$m_2 = \frac{1}{n}\Sigma x_i^2$$

(1) $\hat{\lambda} = \bar{x}$

(2) $\hat{\lambda}^2 + \hat{\lambda} - \frac{1}{n}\sum x_{\hat{\lambda}}^2 = 0 \Rightarrow \hat{\lambda} = -\frac{1}{2} + \left[\frac{1}{4} + \frac{1}{2}3x_i^2\right]^{1/2}$

$\hat{\lambda}$ is not unique, using method of moments.

Next Week:

Estimation Theory 02:
UMVE & Bayes

Have a good day!