Stochastic Processes

Week 03 (Version 2.0)

Ergodic Stochastic Processes

Stochastic Analysis of LTI Systems

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Outline of Week 03 Lectures

- Ergodic Stochastic Processes
- Stochastic Analysis of LTI Systems
- Power Spectrum

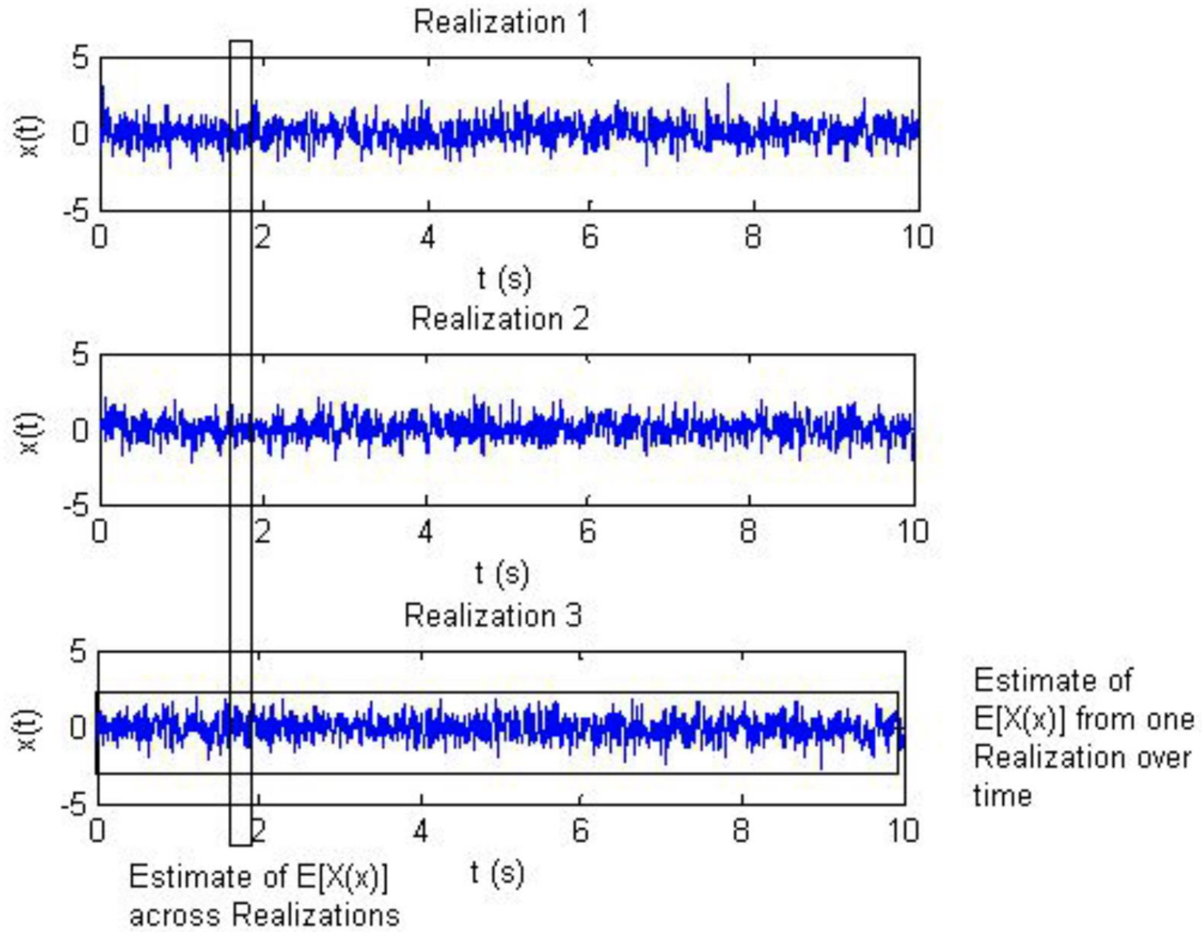
Ergodicity

- A random process X(t) is ergodic if all of its statistics can be determined from a sample function (sample path) of the process.

- That is, the ensemble averages equal the corresponding time averages with probability one.

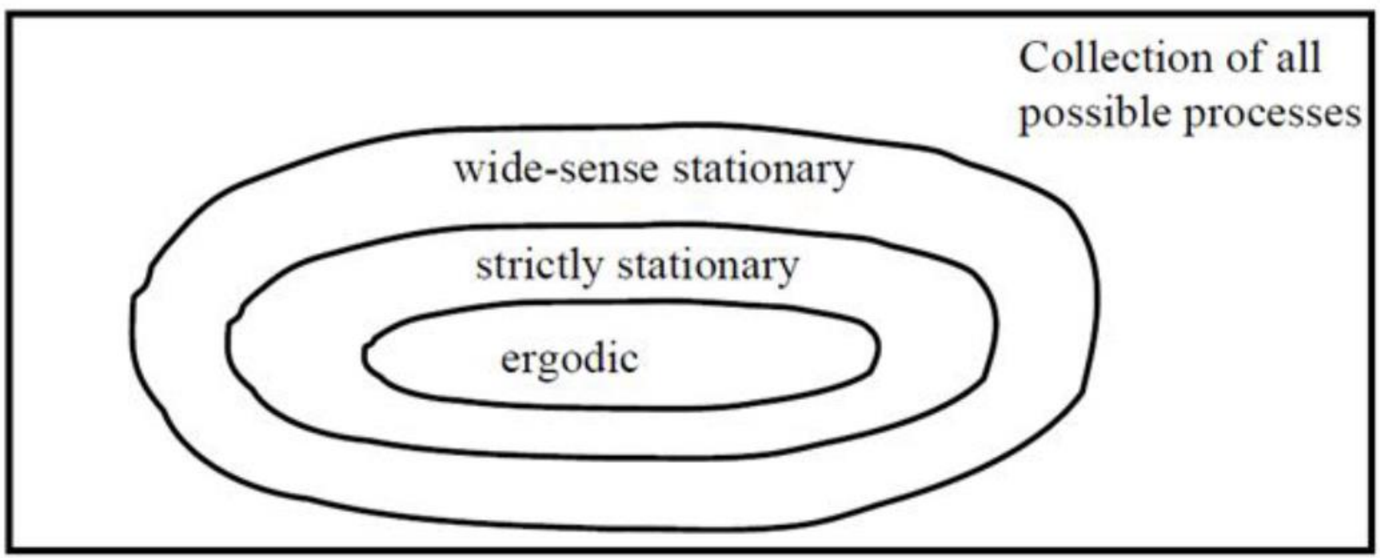
Ergodicity illustrated

- Statistics can be determined by time averaging of one realization (one sample path).



Ergodicity and stationarity

- Wide-sense stationary (WSS): Mean is constant over time and autocorrelation is a function of time difference.
- Strictly stationary (SSS): All statistics are constant over time.
- In general an ergodic process is SSS and WSS.



Weak forms of ergodicity

- The complete statistics is often difficult to estimate so we are often only interested in:
  - ✓ Ergodicity in mean
  - ✓ Ergodicity in autocorrelation

Ergodicity in mean

- A random process is ergodic in mean if E(X(t)) equals the time average of sample function (Realization):

$$E(X(t)) = \langle x(t) \rangle$$

- Where <.> denotes time-averaging:

$$\langle x(t) \rangle = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t)dt$$

- Necessary and sufficient condition:

  X(t+τ) and X(t) must become independent as τ approaches ∞.

Example 1-a

- Example of ergodic in mean:
  $X(t) = a \cos(\omega_0 t + \theta)$
  - Where: θ is a random variable U $[0,2\pi]$, t is the time index, a and ω_0 are constant variables is a WSS process with mean zero.
  - Mean is independent of random variable θ.
- Example of NOT ergodic in mean:
  $X(t) = a \cos(\omega_0 t + \theta) + c_r$
  - Where: θ is a random variable $U(0,2\pi)$, c_r is a random variable, t is the time index, a and ω_0 are constant variables.
  - Mean is not independent of the random variable c_r.

Example 1-b

- Example of ergodic in mean:

  X(t) = a sin($\omega_r\, t + \theta$)

  - Where :
    - ✓ θ is a uniform random variable on $[-\pi, \pi]$
    - ✓ a and ω_r are constant variables
  - Mean is independent of t (is zero)
  - Time average goes to zero ($T \to \infty$)
- Example of NOT ergodic in mean:

  X(t) = a sin($\omega_r t + \ \theta$) + c_r

  - Where :
    - ✓ θ and c_r are random variables
    - ✓ θ is a uniform random variable on $[-\pi, \pi]$
    - ✓ a and ω_r are constant variables
  - Mean is independent of t and c_r
  - But time average doesn't converge in mean squared error ($var(c_r) > 0$) to the mean.

Example 2

Let C be a random variable (RV),
Let X(t) = C be a random process, with mean η_C,
Is X(t) mean ergodic?

Ensemble Average: $E[X(t)] = E[C] = \eta_C$
Time Average: $\eta_T = \frac{1}{2T}\int_{-T}^{T} X(t)dt = \frac{1}{2T}\int_{-T}^{T} Cdt = C$

Time Average is not equal to ensemble average, hence X(t) is not mean ergodic. We can also check the variance of X(t):
$\rightarrow \lim_{T\to\infty} E[(\eta_T - \eta_c)^2] = \lim_{T\to\infty} E[(C - \eta_c)^2] = var(C) > 0$

Ergodicity in autocorrelation

- Ergodic in autocorrelation implies that the autocorrelation can be found by time averaging a single realization:

$$R_{xx}(\boldsymbol{\tau}) = \langle x(t+\boldsymbol{\tau})x(t) \rangle$$

- Where:

$$\langle x(t+\boldsymbol{\tau})x(t) \rangle = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t+\boldsymbol{\tau})x(t)\,dt$$

- Necessary and sufficient condition:

  $x(t+\boldsymbol{\tau})x(t)$ and $x(t+\boldsymbol{\tau}+a)x(t+a)$ must become independent as a approaches ∞.

Example 3

- A random process X(t) is defined as:

$$X(t) = A\cos(2\pi f_c t + \theta)$$

  - ✓ Where A and f_c are constants, and θ is a random variable uniformly distributed over the interval $[0, 2\pi]$
  - ✓ We have seen that the autocorrelation of X(t) is:

$$R_{xx}(\tau) = \frac{A^2}{2}\cos(2\pi f_c \tau) \quad \text{(I)}$$

  - ✓ What is the autocorrelation of a sample function?

Example 3 continued

- The time averaged autocorrelation of the sample function:

$$X(t) = A\cos(2\pi f_c t + \theta)$$

$$\langle X(t+\tau)X(t)\rangle$$

$$= \lim_{T\to\infty} \frac{A^2}{2T}\int_{-T}^{T} \cos[2\pi f_c(t+\tau) + \theta]\cos(2\pi f_c t + \theta)dt$$

$$= \lim_{T\to\infty} \frac{A^2}{4T}\int_{-T}^{T} [\cos(2\pi f_c \tau) + \cos(4\pi f_c t + 2\pi f_c \tau + 2\theta)]dt$$

$$= \frac{A^2}{2}\cos(2\pi f_c \tau) \qquad \text{(II)}$$

- note that: $\cos a \cos b = \frac{1}{2}(\cos(a-b) + \cos(a+b))$
- From I & II we conclude that X(t) is ergodic in autocorrelation

Example 4

$$X(t) \underset{\text{}}{\overset{\text{R.P.}}{\sim}} \text{W.S.S}$$

$$E[X(t)] = 0$$
$$R_{XX}(\tau) = e^{-|\tau|}$$

$$A \underset{\text{R.V.}}{\sim} N(0, 1)$$

$$A \perp\!\!\!\perp X(t)$$

Let $Y(t) = X(t) + A$

Is Y(t) mean ergodic?

Example 4 continued

$$Y(t) = X(t) + A$$

$$E[Y(t)] = E[X(t) + A] = E[X(t)] + E[A] = 0 + 0 = 0$$

$$R_{YY}(t,s) = E[X(t)X(s)] + E[A^2] + 2E[X(t)A] = e^{-|t-s|} + 1$$

mean-ergodicity:

$$E\left[\left(\frac{1}{2T}\int_{-T}^{T} Y(t)dt - \mu_Y\right)^2\right] = \left(\frac{1}{2T}\right)^2 E\left[\iint_{-T}^{T} Y(t)Y(s)\,dtds\right]$$

$$= \left(\frac{1}{2T}\right)^2 \left[\iint_{-T}^{T} E[Y(t)Y(s)]\,dtds\right]$$

not mean ergodic

$$= \left(\frac{1}{2T}\right)^2 \left[\iint_{-T}^{T} e^{-|t-s|} + 1\ \,dtds\right] \geq \left(\frac{1}{2T}\right)^2 \left[\iint_{-T}^{T} 1\,dtds\right] = 1 > 0$$

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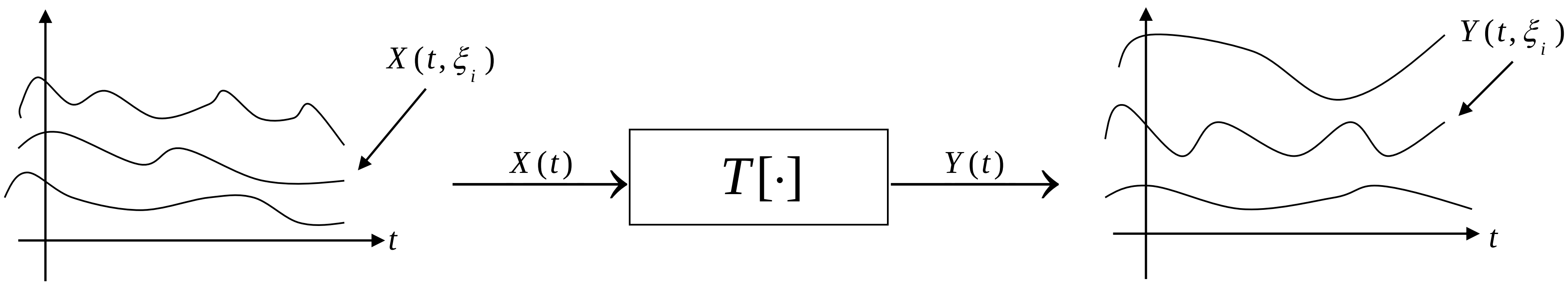
- Stochastic Analysis of LTI Systems
- Power Spectrum

Systems with Stochastic Inputs

A deterministic system transforms each input waveform $X(t, \xi_i)$ into an output waveform $Y(t, \xi_i) = T[X(t, \xi_i)]$ by operating only on the time variable *t*. Thus a set of realizations at the input corresponding to a process $X(t)$ generates a new set of realizations $\{Y(t, \xi)\}$ at the output associated with a new process $Y(t)$.



Our goal is to study the output process statistics in terms of the input process statistics and the system function.

Deterministic Systems

Memoryless Systems

$$Y(t) = g[X(t)]$$

Systems with Memory

Time-varying systems

Time-Invariant systems

Linear systems

$$Y(t) = L[X(t)]$$

Linear-Time Invariant (LTI) systems

$$X(t) \longrightarrow \boxed{h(t)} \longrightarrow Y(t) = \int_{-\infty}^{+\infty} h(t-\tau) X(\tau) d\tau$$

$$= \int_{-\infty}^{+\infty} h(\tau) X(t-\tau) d\tau.$$

LTI system

Memoryless Systems

The output $Y(t)$ in this case depends only on the present value of the input $X(t)$. i.e.;

$$Y(t) = g\{X(t)\}$$

Strict-sense stationary input	→ Memoryless system →	Strict-sense stationary output.
Wide-sense stationary input	→ Memoryless system →	Need *not* be stationary in any sense.
$X(t)$ stationary Gaussian with $R_{XX}(\tau)$	→ Memoryless system →	$Y(t)$ stationary, but *not* Gaussian with $R_{XY}(\tau) = \eta R_{XX}(\tau)$.

Linear Systems: $L[\cdot]$ represents a linear system if

$$L\{a_1 X(t_1) + a_2 X(t_2)\} = a_1 L\{X(t_1)\} + a_2 L\{X(t_2)\}.$$

Let

$$Y(t) = L\{X(t)\}$$

represent the output of a linear system.

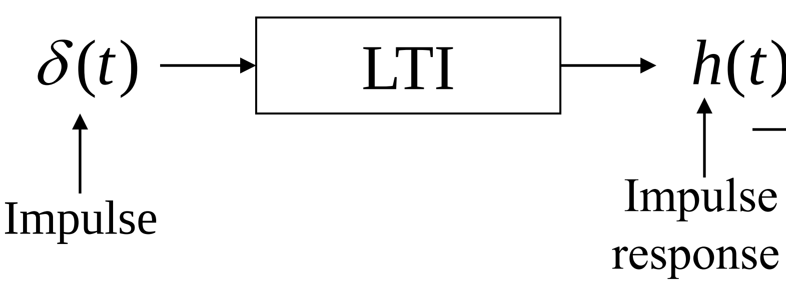
Time-Invariant System: $L[\cdot]$ represents a time-invariant system if

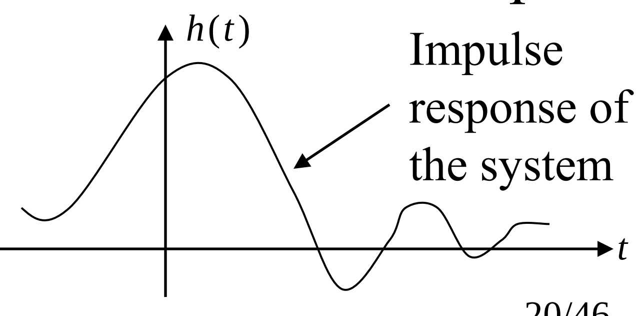
$$Y(t) = L\{X(t)\} \Rightarrow L\{X(t - t_0)\} = Y(t - t_0)$$

i.e., shift in the input results in the same shift in the output.

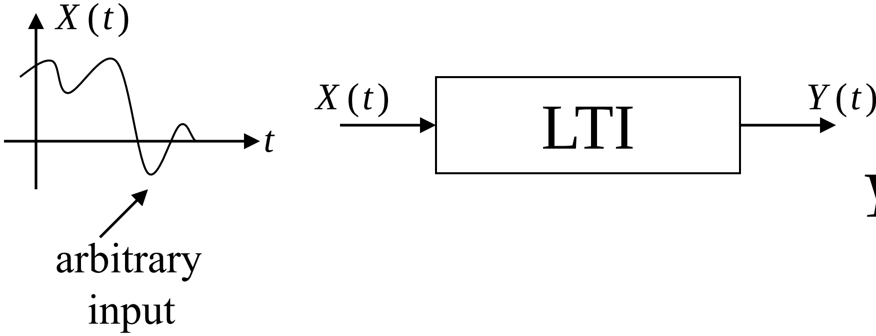
If $L[\cdot]$ satisfies above equations, then it corresponds to a linear time-invariant (LTI) system.

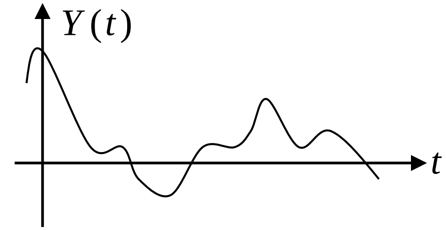
LTI systems can be uniquely represented in terms of their output to a input delta function:





then





$$Y(t) = \int_{-\infty}^{+\infty} h(t-\tau)X(\tau)d\tau$$
$$= \int_{-\infty}^{+\infty} h(\tau)X(t-\tau)d\tau$$

We can express $X(t)$ as:

$$X(t) = \int_{-\infty}^{+\infty} X(\tau)\delta(t-\tau)d\tau$$

But $Y(t) = L\{X(t)\}$. Then:

$$Y(t) = L\{X(t)\} = L\{\int_{-\infty}^{+\infty} X(\tau)\delta(t-\tau)d\tau\}$$

$$= \int_{-\infty}^{+\infty} L\{X(\tau)\delta(t-\tau)d\tau\}$$

By Linearity

$$= \int_{-\infty}^{+\infty} X(\tau)L\{\delta(t-\tau)\}d\tau$$

By Time-invariance

$$= \int_{-\infty}^{+\infty} X(\tau)h(t-\tau)d\tau = \int_{-\infty}^{+\infty} h(\tau)X(t-\tau)d\tau.$$

Output Statistics: The mean of the output process is given by

$$\mu_Y(t) = E\{Y(t)\} = \int_{-\infty}^{+\infty} E\{X(\tau)h(t-\tau)d\tau\}$$

$$= \int_{-\infty}^{+\infty} \mu_X(\tau)h(t-\tau)d\tau = \mu_X(t) * h(t).$$

Similarly the cross-correlation function between the input and output processes is given by:

$$R_{XY}(t_1,t_2) = E\{X(t_1)Y^*(t_2)\}$$

$$= E\{X(t_1)\int_{-\infty}^{+\infty} X^*(t_2-\alpha)h^*(\alpha)d\alpha\}$$

$$= \int_{-\infty}^{+\infty} E\{X(t_1)X^*(t_2-\alpha)\}h^*(\alpha)d\alpha$$

$$= \int_{-\infty}^{+\infty} R_{XX}(t_1,t_2-\alpha)h^*(\alpha)d\alpha$$

$$= R_{XX}(t_1,t_2) * h^*(t_2).$$

Finally the output autocorrelation function is given by:

$$
\begin{aligned}
R_{YY}(t_1,t_2) &= E\{Y(t_1)Y^*(t_2)\} \\
&= E\{\int_{-\infty}^{+\infty} X(t_1-\beta)h(\beta)d\beta\, Y^*(t_2)\} \\
&= \int_{-\infty}^{+\infty} E\{X(t_1-\beta)Y^*(t_2)\}h(\beta)d\beta \\
&= \int_{-\infty}^{+\infty} R_{XY}(t_1-\beta,t_2)h(\beta)d\beta \\
&= R_{XY}(t_1,t_2) * h(t_1),
\end{aligned}
$$

or

$$R_{YY}(t_1,t_2) = R_{XX}(t_1,t_2) * h^*(t_2) * h(t_1).$$

$$\mu_X(t) \longrightarrow \boxed{h(t)} \longrightarrow \mu_Y(t)$$

$$R_{XX}(t_1,t_2) \longrightarrow \boxed{h^*(t_2)} \xrightarrow{R_{XY}(t_1,t_2)} \boxed{h(t_1)} \longrightarrow R_{YY}(t_1,t_2)$$

In particular if $X(t)$ is wide-sense stationary, then we have $\mu_X(t) = \mu_X$
Then:

$$\mu_Y(t) = \mu_X \int_{-\infty}^{+\infty} h(\tau)d\tau = \mu_X c, \quad a\ constant.$$

Also $R_{XX}(t_1, t_2) = R_{XX}(t_1 - t_2)$, and:

$$R_{XY}(t_1, t_2) = \int_{-\infty}^{+\infty} R_{XX}(t_1 - t_2 + \alpha) h^*(\alpha) d\alpha$$
$$= R_{XX}(\tau) * h^*(-\tau) \stackrel{\Delta}{=} R_{XY}(\tau), \quad \tau = t_1 - t_2.$$

Thus $X(t)$ and $Y(t)$ are jointly w.s.s., and the output autocorrelation simplifies to:

$$R_{YY}(t_1, t_2) = \int_{-\infty}^{+\infty} R_{XY}(t_1 - \beta - t_2) h(\beta) d\beta, \qquad \tau = t_1 - t_2$$
$$= R_{XY}(\tau) * h(\tau) = R_{YY}(\tau).$$

And we obtain:

$$R_{YY}(\tau) = R_{XX}(\tau) * h^*(-\tau) * h(\tau).$$

The output process is also wide-sense stationary.
This gives rise to the following representation.

$X(t)$ wide-sense stationary process $\longrightarrow$ LTI system $h(t)$ $\longrightarrow$ $Y(t)$ wide-sense stationary process.

(a)

$X(t)$ strict-sense stationary process $\longrightarrow$ LTI system $h(t)$ $\longrightarrow$ $Y(t)$ strict-sense stationary process

(b)

$X(t)$ Gaussian process (also stationary) $\longrightarrow$ Linear system $\longrightarrow$ $Y(t)$ Gaussian process (also stationary)

(c)

White Noise Process

$W(t)$ is said to be a white noise process if:

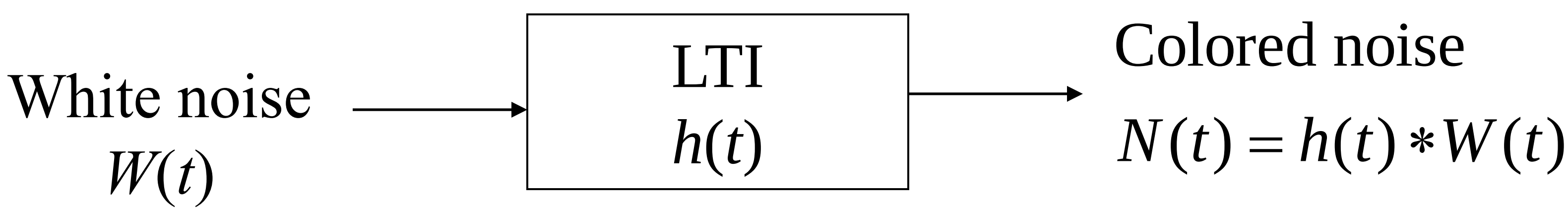
$$R_{ww}(t_1, t_2) = q(t_1)\delta(t_1 - t_2),$$

i.e., $E[W(t_1)\ W^*(t_2)] = 0$ unless $t_1 = t_2$.

$W(t)$ is said to be wide-sense stationary (w.s.s) white noise if $E[W(t)]$ = constant, and:

$$R_{ww}(t_1, t_2) = q\delta(t_1 - t_2) = q\delta(\tau).$$

If $W(t)$ is also a Gaussian process (white Gaussian process), then all of its samples are independent random variables.



For w.s.s. white noise input $W(t)$, we have:

$$E[N(t)] = \mu_W \int_{-\infty}^{+\infty} h(\tau) d\tau, \quad a\ constant$$

and:

$$\begin{aligned} R_{nn}(\tau) &= q\delta(\tau) * h^*(-\tau) * h(\tau) \\ &= qh^*(-\tau) * h(\tau) = q\rho(\tau) \end{aligned}$$

where:

$$\rho(\tau) = h(\tau) * h^*(-\tau) = \int_{-\infty}^{+\infty} h(\alpha) h^*(\tau - \alpha) d\alpha .$$

Thus the output of a white noise process through an LTI system represents a (colored) noise process.

Note: White noise need not be Gaussian.

"White" and "Gaussian" are two different concepts!

Discrete Time Stochastic Processes

A discrete time stochastic process $X_n = X(nT)$ is a sequence of random variables. The mean, autocorrelation and auto-covariance functions of a discrete-time process are gives by:

$$\mu_n = E\{X(nT)\}$$

$$R(n_1, n_2) = E\{X(n_1 T) X^*(n_2 T)\}$$

and

$$C(n_1, n_2) = R(n_1, n_2) - \mu_{n_1} \mu_{n_2}^*$$

respectively. As before strict sense stationarity and wide-sense stationarity definitions apply here.
For example, $X(nT)$ is wide sense stationary if:

$$E\{X(nT)\} = \mu, \quad a\ constant$$

and

$$E[X\{(k+n)T\} X^*\{(k)T\}] = R(n) = r_n \triangleq r_{-n}^*$$

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Power Spectrum

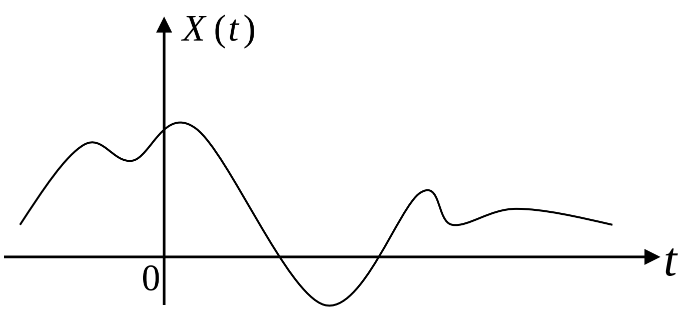
For a deterministic signal x(t), the spectrum is well defined: If $X(\omega)$ represents its Fourier transform, i.e., if;

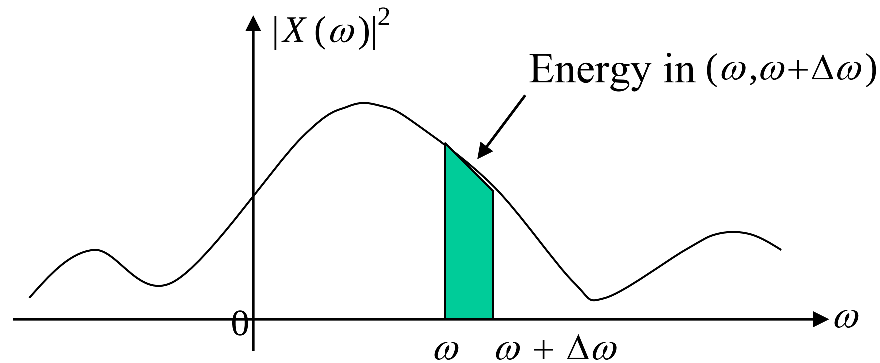
$$X(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt,$$

then $|X(\omega)|^2$ represents its energy spectrum. This follows from Parseval's theorem since the signal energy is given by:

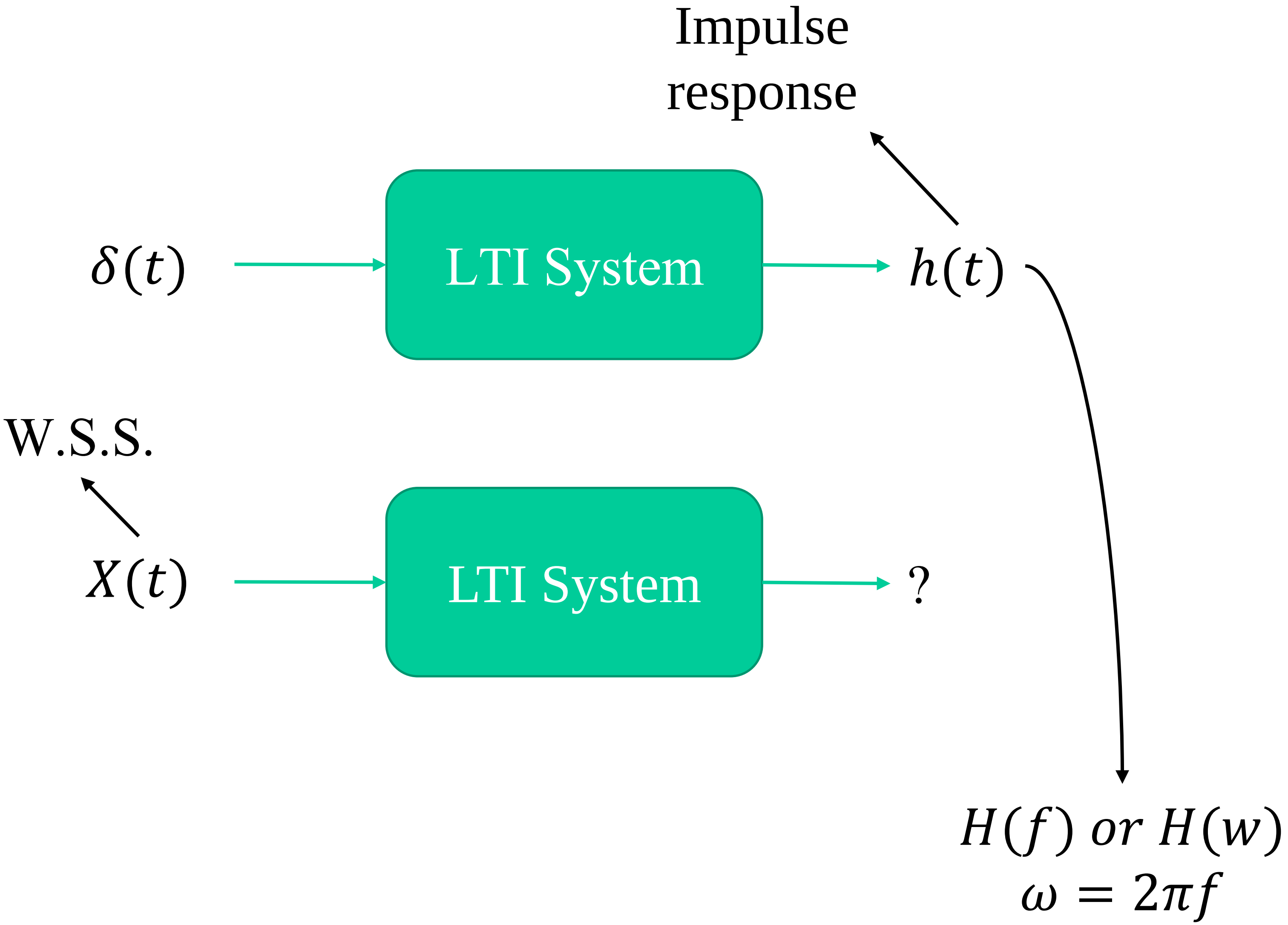
$$\int_{-\infty}^{+\infty} x^2(t) dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X(\omega)|^2 \, d\omega = E.$$

Thus $|X(\omega)|^2 \Delta\omega$ represents the signal energy in the band $(\omega, \omega + \Delta\omega)$





Power Spectrum



However for stochastic processes, a direct application of X(ω) generates a sequence of random variables for every ω. Moreover, for a stochastic process, $E\{|X(t)|^2\}$ represents the ensemble average power (instantaneous energy) at the instant t.

To obtain the spectral distribution of power versus frequency for stochastic processes, it is best to avoid infinite intervals to begin with, and start with a finite interval $(-T, \; T)$. Formally, partial Fourier transform of a process $X(t)$ based on $(-T, \; T)$ is given by:

$$X_T(\omega) = \int_{-T}^{T} X(t) e^{-j\omega t} dt$$

so that:

$$\frac{|X_T(\omega)|^2}{2T} = \frac{1}{2T} \left| \int_{-T}^{T} X(t) e^{-j\omega t} dt \right|^2$$

represents the power distribution associated with that realization on $(-T, \; T)$. Notice that the above represents a RV for every ω, and its ensemble average gives, the average power distribution on $(-T, \; T)$. Thus:

$$P_T(\omega) = E\left\{\frac{|X_T(\omega)|^2}{2T}\right\} = \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} E\{X(t_1)X^*(t_2)\}e^{-j\omega(t_1-t_2)}dt_1dt_2$$

$$= \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_{XX}(t_1,t_2)e^{-j\omega(t_1-t_2)}dt_1dt_2$$

represents the power distribution of $X(t)$ on $(-T, T)$.

Thus if $X(t)$ is assumed to be w.s.s, then $R_{XX}(t_1,t_2) = R_{XX}(t_1 - t_2)$ and:

$$P_T(\omega) = \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_{XX}(t_1 - t_2)e^{-j\omega(t_1-t_2)}dt_1dt_2.$$

Let $\tau = t_1 - t_2$, we get:

$$P_T(\omega) = \frac{1}{2T}\int_{-2T}^{2T} R_{XX}(\tau)e^{-j\omega\tau}(2T - |\tau|)d\tau$$

$$= \int_{-2T}^{2T} R_{XX}(\tau)e^{-j\omega\tau}\left(1 - \tfrac{|\tau|}{2T}\right)d\tau \;\geq\; 0$$

to be the power distribution of the w.s.s. process $X(t)$ based on $(-T, T)$. Finally letting $T \to \infty$, we obtain:

$$S_{XX}(\omega) = \lim_{T\to\infty} P_T(\omega) = \int_{-\infty}^{+\infty} R_{XX}(\tau) e^{-j\omega\tau} d\tau \geq 0$$

to be the *power spectral density* of the w.s.s process $X(t)$. Notice that

$$R_{XX}(\omega) \xleftrightarrow{\text{F·T}} S_{XX}(\omega) \geq 0.$$

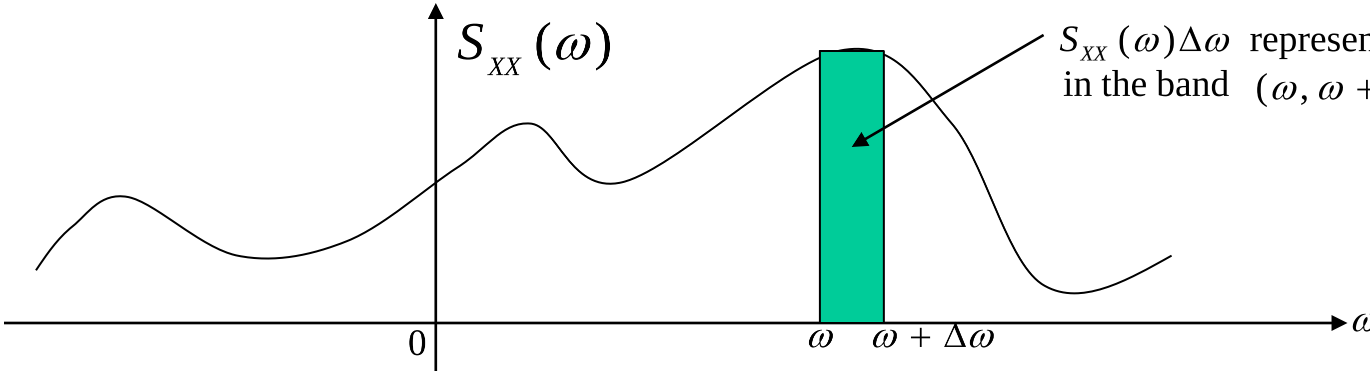
i.e., the autocorrelation function and the power spectrum of a w.s.s Process form a Fourier transform pair, a relation known as the **Wiener-Khinchin Theorem**. The inverse formula gives:

$$R_{XX}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XX}(\omega) e^{j\omega\tau} d\omega$$

and in particular for $\tau = 0$, we get:

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XX}(\omega) d\omega = R_{XX}(0) = E\{|X(t)|^2\} = P, \quad \textit{the total power}.$$

The area under $S_{XX}(\omega)$ represents the total power of the process $X(t)$, and hence $S_{XX}(\omega)$ truly represents the power spectrum.



The nonnegative-definiteness property of the autocorrelation function translates into the "nonnegative" property for its Fourier transform (power spectrum), since:

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j^* R_{XX}(t_i - t_j) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j^* \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{XX}(\omega) e^{j\omega(t_i - t_j)} d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{XX}(\omega) \left| \sum_{i=1}^{n} a_i e^{j\omega t_i} \right|^2 d\omega \geq 0.$$

It follows that:

$$R_{XX}(\tau)\ \textit{nonnegative-definite} \quad \Leftrightarrow \quad S_{XX}(\omega) \geq 0.$$

If $X(t)$ is a real w.s.s process, then $R_{XX}(\tau) = R_{XX}(-\tau)$ so that

$$
\begin{aligned}
S_{XX}(\omega) &= \int_{-\infty}^{+\infty} R_{XX}(\tau) e^{-j\omega\tau} d\tau \\
&= \int_{-\infty}^{+\infty} R_{XX}(\tau) \cos \omega\tau \, d\tau \\
&= 2\int_{0}^{\infty} R_{XX}(\tau) \cos \omega\tau \, d\tau = S_{XX}(-\omega) \geq 0
\end{aligned}
$$

so that the power spectrum is an even function, (in addition to being real and nonnegative).

Power Spectra and LTI Systems

If a w.s.s process $X(t)$ with autocorrelation function $R_{XX}(\tau) \leftrightarrow S_{XX}(\omega) \geq 0$ is applied to a linear system with impulse response $h(t)$, then the cross correlation function $R_{XY}(\tau)$ and the output autocorrelation function $R_{YY}(\tau)$:

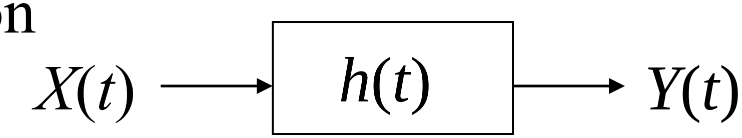


Fig 18.3

$$R_{XY}(\tau) = R_{XX}(\tau) * h^*(-\tau), \quad R_{YY}(\tau) = R_{XX}(\tau) * h^*(-\tau) * h(\tau).$$

Then:

$$f(t) \leftrightarrow F(\omega), \quad g(t) \leftrightarrow G(\omega)$$

Since:

$$f(t) * g(t) \leftrightarrow F(\omega)G(\omega)$$

$$\mathbf{F}\,\{f(t) * g(t)\} = \int_{-\infty}^{+\infty} f(t) * g(t) e^{-j\omega t} dt$$

$$\mathbf{F}\ \{f(t)*g(t)\}=\int_{-\infty}^{+\infty}\left\{\int_{-\infty}^{+\infty} f(\tau)g(t-\tau)d\tau\right\}e^{-j\omega t}dt$$

$$=\int_{-\infty}^{+\infty} f(\tau)e^{-j\omega\tau}d\tau\ \int_{-\infty}^{+\infty} g(t-\tau)e^{-j\omega(t-\tau)}d(t-\tau)$$

$$=F(\omega)G(\omega).$$

Then we get:

$$S_{XY}(\omega)=\mathbf{F}\ \{R_{XX}(\omega)*h^{*}(-\tau)\}=S_{XX}(\omega)H^{*}(\omega)$$

Since:

$$\int_{-\infty}^{+\infty} h^{*}(-\tau)e^{-j\omega\tau}d\tau=\left(\int_{-\infty}^{+\infty} h(t)e^{-j\omega t}dt\right)^{*}=H^{*}(\omega),$$

Where:

$$H(\omega)=\int_{-\infty}^{+\infty} h(t)e^{-j\omega t}dt$$

represents the transfer function of the system, and:

$$S_{YY}(\omega)=\mathbf{F}\ \{R_{YY}(\tau)\}=S_{XY}(\omega)H(\omega)$$

$$=S_{XX}(\omega)\,|\,H(\omega)\,|^{2}\ .$$

The cross spectrum need not be real or nonnegative;
However the output power spectrum is real and nonnegative and is related to the input spectrum and the system transfer function can be used for system identification as well.

W.S.S White Noise Process: If $W(t)$ is a w.s.s white noise process, then:

$$R_{WW}(\tau) = q\delta(\tau) \quad \Rightarrow \quad S_{WW}(\omega) = q.$$

Thus the spectrum of a white noise process is flat, thus justifying its name. Notice that a white noise process is unrealizable since its total power is indeterminate.
If the input to an unknown system is
a white noise process, then the output spectrum is given by:

$$S_{YY}(\omega) = q\,|\,H(\omega)\,|^2$$

Notice that the output spectrum captures the system transfer function characteristics entirely, and for rational systems may be used to determine the pole/zero locations of the underlying system.

Example: A w.s.s white noise process $W(t)$ is passed through a low pass filter (LPF) with bandwidth $B/2$. Find the autocorrelation function of the output process.

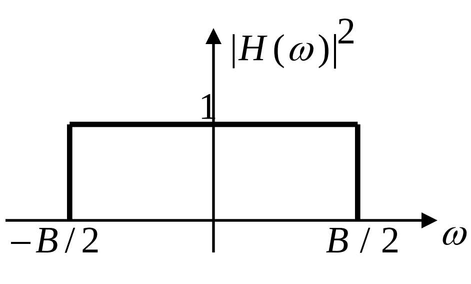
Solution: Let $X(t)$ represent the output of the LPF. Then:

$$S_{XX}(\omega) = q\,|H(\omega)|^2 = \begin{cases} q, & |\omega| \le B/2 \\ 0, & |\omega| > B/2 \end{cases}.$$

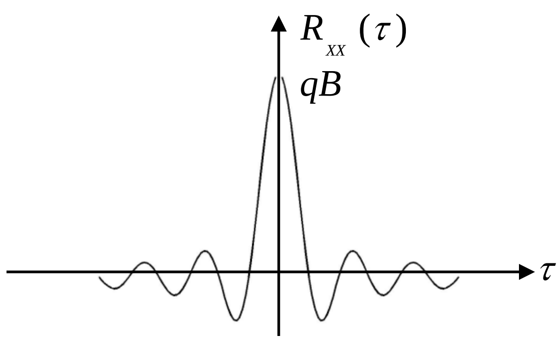
Inverse transform of $S_{XX}(\omega)$ gives the output autocorrelation function to be:

$$R_{XX}(\tau) = \int_{-B/2}^{B/2} S_{XX}(\omega) e^{j\omega\tau} d\omega = q\int_{-B/2}^{B/2} e^{j\omega\tau} d\omega$$

$$= qB\,\frac{\sin(B\tau/2)}{(B\tau/2)} = qB\,\mathrm{sinc}(B\tau/2)$$



(a) LPF



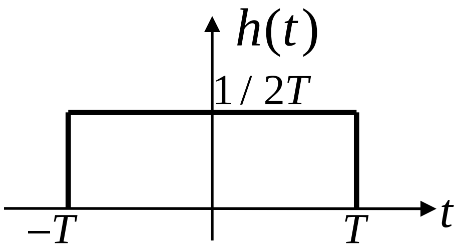
(b)

Example: Let:

$$Y(t) = \frac{1}{2T}\int_{t-T}^{t+T} X(\tau)d\tau$$

represent a "smoothing" operation using a moving window on the input process $X(t)$. Find the spectrum of the output $Y(t)$ in term of $X(t)$.

Solution: If we define an LTI system with impulse response $h(t)$, then in term of $h(t)$:



$$Y(t) = \int_{-\infty}^{+\infty} h(t-\tau)X(\tau)d\tau = h(t) * X(t)$$

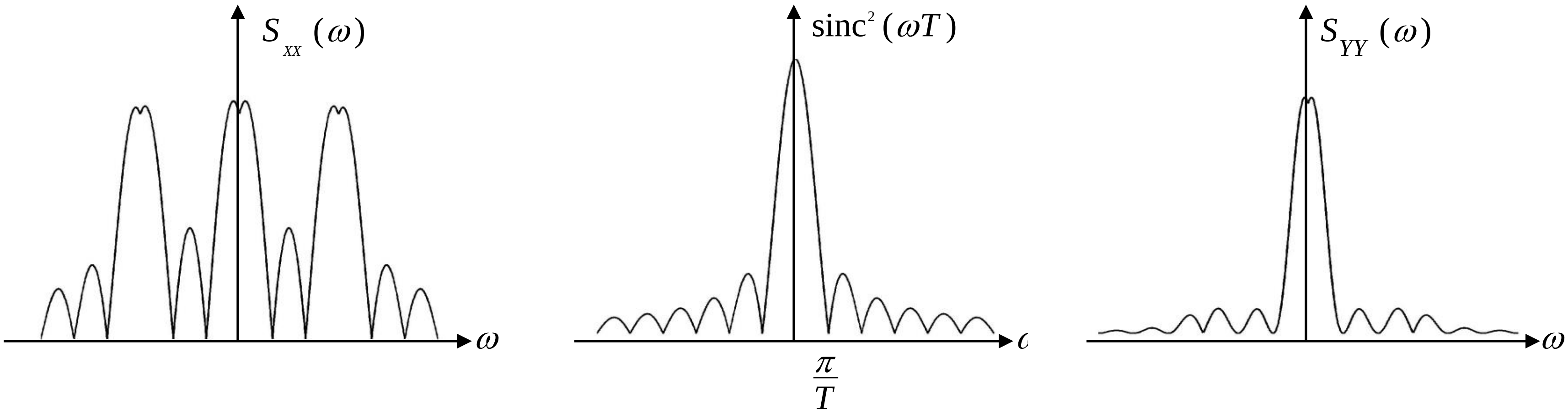
Here

$$S_{YY}(\omega) = S_{XX}(\omega)\,|\,H(\omega)\,|^2\,.$$

$$H(\omega) = \int_{-T}^{+T} \tfrac{1}{2T} e^{-j\omega t} dt = \text{sinc}(\omega T)$$

so that:

$$S_{YY}(\omega) = S_{XX}(\omega)\,\text{sinc}^2(\omega T).$$



Notice that the effect of the smoothing operation is to suppress the high frequency components in the input (beyond π / T), and the equivalent linear system acts as a low-pass filter (continuous-time moving average) with bandwidth $2\pi / T$ in this case.

Discrete – Time Processes

For discrete-time w.s.s stochastic processes $X(nT)$ with autocorrelation sequence $\{r_k\}_{-\infty}^{+\infty}$, (proceeding as above) or formally defining a continuous time process $X(t) = \sum_n X(nT)\delta(t-nT)$, we get the corresponding autocorrelation function to be:

$$R_{XX}(\tau) = \sum_{k=-\infty}^{+\infty} r_k \delta(\tau - kT).$$

Its Fourier transform is given by:

$$S_{XX}(\omega) = \sum_{k=-\infty}^{+\infty} r_k e^{-j\omega T} \geq 0,$$

and it defines the power spectrum of the discrete-time process $X(nT)$.

$$S_{XX}(\omega) = S_{XX}(\omega + 2\pi / T)$$

so that $S_{XX}(\omega)$ is a periodic function with period

$$2B = \frac{2\pi}{T}.$$

This gives the inverse relation:

$$r_k = \frac{1}{2B}\int_{-B}^{B} S_{XX}(\omega)e^{jk\omega T}\,d\omega$$

and:

$$r_0 = E\{|X(nT)|^2\} = \frac{1}{2B}\int_{-B}^{B} S_{XX}(\omega)\,d\omega$$

represents the total power of the discrete-time process $X(nT)$. The input-output relations for discrete-time system $h(nT)$ translate into:

$$S_{XY}(\omega) = S_{XX}(\omega)H^*(e^{j\omega})$$

And:

$$S_{YY}(\omega) = S_{XX}(\omega)\,|H(e^{j\omega})|^2$$

Where:

$$H(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} h(nT)\,e^{-j\omega nT}$$

represents the discrete-time system transfer function.

Summary of LTI Systems with Stochastic Inputs

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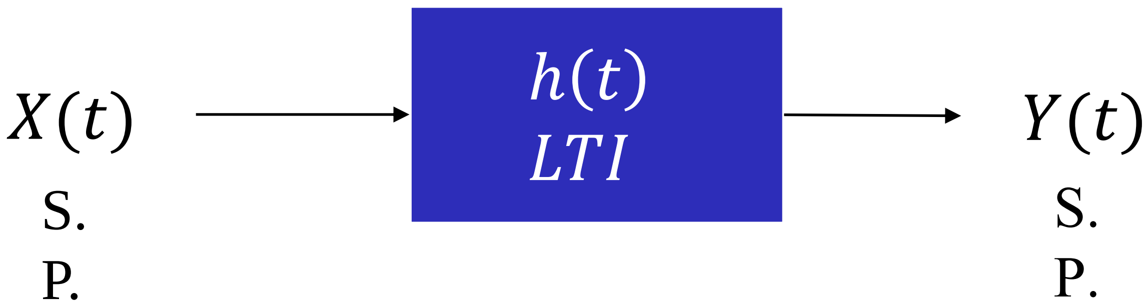
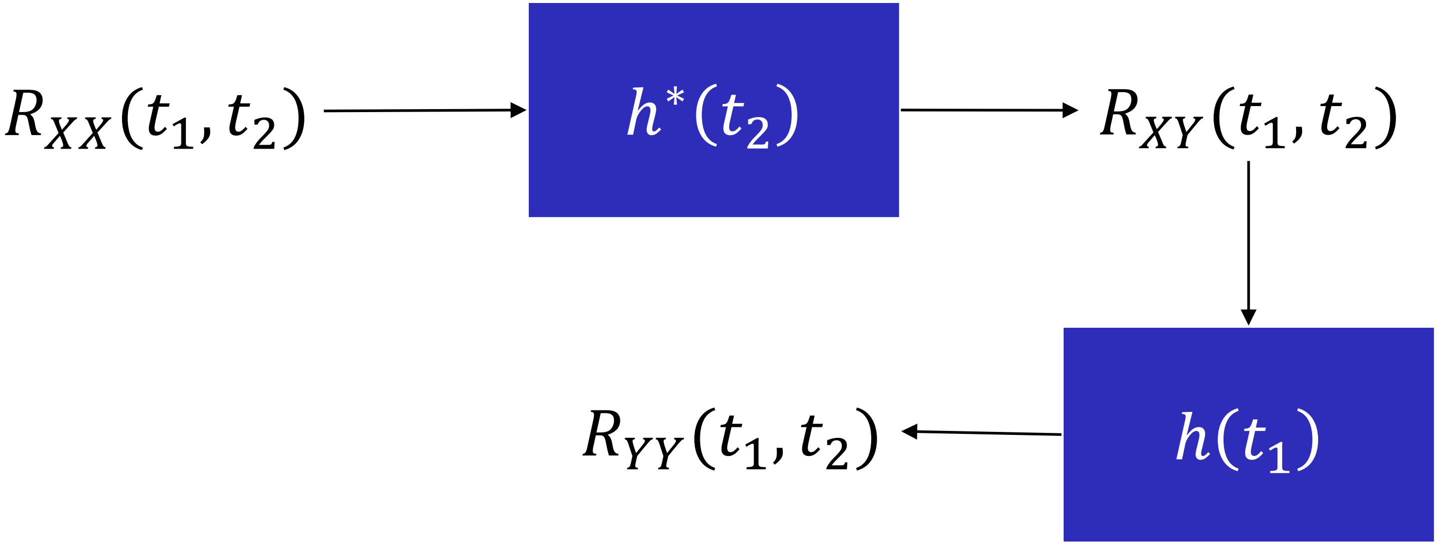


$$\mu_Y(t) = \mu_X(t) * h(t)$$

$$R_{XY}(t_1, t_2) = R_{XX}(t_1, t_2) * h^*(t_2)$$
$$R_{YY}(t_1, t_2) = R_{XY}(t_1, t_2) * h(t_1)$$
$$R_{YY}(t_1, t_2) = R_{XX}(t_1, t_2) * h^*(t_2) * h(t_1)$$

Summary of LTI Systems with Stochastic Inputs



Summary of LTI Systems with WSS Stochastic Inputs

$Let\ X(t)\ be\ a\ WSS\ Stochastic\ Process$(input), h(t) impulse response of an LTI system, and y(t) its output, then:

$$\mu_Y(t) = \mu_X c = constant$$

$$R_{XY}(\tau) = R_{XX}(\tau) * h^*(-\tau)$$
$$R_{YY}(\tau) = R_{XY}(\tau) * h(\tau)$$
$$R_{YY}(\tau) = R_{XX}(\tau) * h^*(-\tau) * h(\tau)$$

$$S_{XX}(\omega) = \mathcal{F}\big(R_{XX}(\tau)\big)$$
$$S_{XY}(\omega) = S_{XX}(\omega)H^*(\omega)$$
$$S_{YY}(\omega) = S_{XY}(\omega)H(\omega)$$
$$S_{YY}(\omega) = S_{XX}(\omega)H^*(\omega)H(\omega) = S_{XX}(\omega)|H(\omega)|^2$$

Next Week:

Poisson Processes
Point Processes

Have a good day!