In the name of GOD.
Sharif University of Technology
Stochastic Processes CE 695 Fall 2022 H.R. Rabiee

Solution Homework 3 (LTI + Ergodicity)

1. Short answer:

   (a) Is sum of two WSS processes always a WSS process?

   (b) Assume a stochastic process such that $X(t) = At + b$. We know that b is a constant and $A \sim Normal(0, 1)$. Is this process mean ergodic?

   (c) We have a stochastic process that is zero mean and $S_X(\omega) = \frac{5}{9+\omega^2}$. Is this process mean ergodic?

   Solution:

   (a) No. For the autocorrlation of the new process we have:

   $$\begin{aligned} R_{X+Y}(t,s) &= \mathbb{E}[(X(t)+Y(t))(X(s)+Y(s))] \\ &= \mathbb{E}[X(t)X(S)] + \mathbb{E}[Y(t)Y(S)] + \mathbb{E}[X(t)X(S)] + \mathbb{E}[X(s)X(t)] \\ &= R_X(t-s) + R_Y(t-s) + 2R_{XY}(s,t) \end{aligned}$$

   (b) No. For autocorrelation of this process we have:

   $$R_X(t,s) = \mathbb{E}[(At+b)(As+b)] = ts\mathbb{E}[A^2] + b(t+s)\mathbb{E}[A] + b^2 = ts + b^2$$

   (c) Yes.

   $$S_x(\omega) = \frac{5}{9+w^2} \xrightarrow{I.F.T} R_x = \frac{5}{6}e^{-3|\tau|}$$
   $$C_x(\tau) = R_x(\tau) - \mu_x = \frac{5}{6}e^{-3|\tau|}$$

   And then:

   $$\begin{aligned} \lim_{T\to\infty} \frac{1}{T}\int_0^T C_x(\tau)d\tau &= \lim_{T\to\infty} \frac{1}{T}\int_0^T \frac{5}{6}e^{-3\tau}d\tau \\ &= \lim_{T\to\infty} \frac{5\left(e^{-3T}-1\right)}{-18T} \\ &= \lim_{T\to\infty} \frac{5}{18T} \\ &= 0 \end{aligned}$$

2. Suppose we have an LTI system with the impulse response like below:

   $$|H(f)| = \begin{cases} \sqrt{1+4\pi^2 f^2} & |f| < 2 \\ 0 & \text{otherwise} \end{cases}$$

Assume X(t) is a zero mean WSS process such that:

$$R_X(\tau) = e^{-|\tau|}$$

If X(t) is the input and Y(t) is the output, calculate below parameters:

$$\mu_Y(t)$$

$$R_Y(\tau)$$

$$\mathbb{E}[Y^2(t)]$$

Solution:
To find $R_Y(\tau)$, we first find $S_Y(f)$.

$$S_Y(f) = S_X(f)|H(f)|^2.$$

From Fourier transform tables, we can see that

$$\begin{aligned} S_X(f) &= \mathcal{F}\left\{e^{-|\tau|}\right\} \\ &= \frac{2}{1+(2\pi f)^2}. \end{aligned}$$

Then, we can find $S_Y(f)$ as

$$\begin{aligned} S_Y(f) &= S_X(f)|H(f)|^2 \\ &= \begin{cases} 2 & |f| < 2 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

We can now find $R_Y(\tau)$ by taking the inverse Fourier transform of $S_Y(f)$.

$$R_Y(\tau) = 8\,\text{sinc}(4\tau),$$

where

$$\text{sinc}(f) = \frac{\sin(\pi f)}{\pi f}.$$

We have

$$E\left[Y(t)^2\right] = R_Y(0) = 8$$

3. Suppose X(t) is a WSS stochastic process and we know that:

$$\mathbb{E}[X(t)] = \mu_X$$

$$R_X(\tau) = e^{-|\tau|}$$

Assume that A is an independent Gaussian random variable such that:

$$A \sim Normal(\mu_A, \sigma_A^2)$$

If we define Y(t) as below:

$$Y(t) = X(t) + A$$

Prove that :

$$\sigma_A^2 = \mu_X^2 \Leftrightarrow Y(t) \text{ is mean ergodic}$$

Solution:

$$\mu_Y(t) = \mathbb{E}[Y(t)] = \mathbb{E}[X(t)] + \mathbb{E}[A] = \mu_X + \mu_A$$
$$\begin{aligned} R_Y(t,s) &= \mathbb{E}[Y(t)Y(s)] = \mathbb{E}[X(t)X(s)] + \mathbb{E}[AX(t)] + \mathbb{E}[AX(s)] + \mathbb{E}[A^2] \\ &= R_X(t-s) + \mathbb{E}[A]\mathbb{E}[X(t)] + \mathbb{E}[A]\mathbb{E}[X(s)] + (\mu_A^2 + \sigma_A^2) \\ &= R_X(t-s) + 2\mu_A\mu_X + \mu_A^2 + \sigma_A^2 \end{aligned}$$

This process is WSS. So we can calculate autocovariance by:

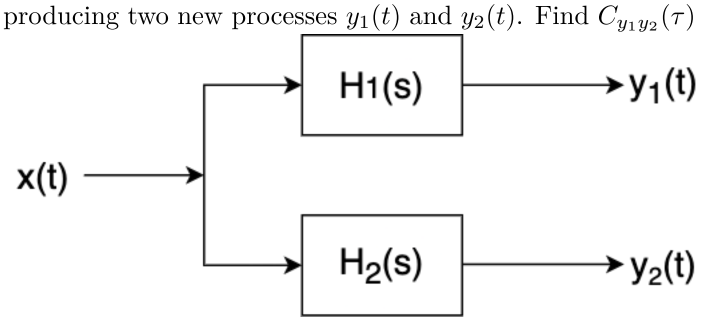
$$\begin{aligned} C_Y(\tau) &= R_Y(\tau) - \mu_Y^2 \\ &= R_X(\tau) + 2\mu_A\mu_X + \mu_A^2 + \sigma_A^2 - (\mu_X + \mu_A)^2 \\ &= R_X(\tau) + \sigma_A^2 - \mu_X^2 \end{aligned}$$

and then:

$$\begin{aligned} \frac{1}{T}\int_0^T C_Y(\tau)d\tau &= \frac{1}{T}\int_0^T R_X(\tau)d\tau + \frac{1}{T}\int_0^T (\sigma_A^2 - \mu_X^2)d\tau \\ &= (\frac{1}{T}\int_0^T e^{-|\tau|}d\tau) + \sigma_A^2 - \mu_X^2 \\ &= \frac{1-e^{-T}}{T} + \sigma_A^2 - \mu_X^2 \\ &\xrightarrow{T\to\infty} 0 + \sigma_A^2 - \mu_X^2 \end{aligned}$$

$$\frac{1}{T}\int_0^T C_Y(\tau)d\tau \xrightarrow{T\to\infty} 0 \Leftrightarrow \sigma_A^2 - \mu_X^2 = 0 \Leftrightarrow \sigma_A^2 = \mu_X^2$$

4. Let $x(t)$ be a real valued, continuous time, zero mean WSS random process with correlation function $R_{xx}(\tau)$ and power spectrum $S_{xx}(\omega)$. Suppose $x(t)$ is the input to two real valued LTI systems as depicted below, producing two new processes $y_1(t)$ and $y_2(t)$. Find $C_{y_1y_2}(\tau)$ and $S_{y_1y_2}(\omega)$



Solution:

$$\begin{aligned}
R_{y_1y_2}(\tau) &= E\left[y_1(t+\tau)\cdot y_2(t)\right] \\
&= E\left[\left(\int_{-\infty}^{+\infty} x(t+\tau-a)\cdot h_1(a)da\right) y_2(t)\right] \\
&= \int_{-\infty}^{+\infty} E\left[x(t+\tau-a)\cdot h_1(a)y_2(t)\right] da \\
&= \int_{-\infty}^{+\infty} R_{xy_2}(\tau-a)\cdot h_1(a)da \\
&= R_{xy_2}(\tau) * h_1(\tau) \\
R_{xy_2}(\tau) &= E\left[x(t+\tau)\cdot y_2(t)\right] \\
&= E\left[x(t+\tau)\left(\int_{-\infty}^{+\infty} x(t-a)\cdot h_2(a)da\right)\right] \\
&= \int_{-\infty}^{+\infty} E\left[x(t+\tau)\cdot x(t-a)\cdot h_2(a)\right] da \\
&= \int_{-\infty}^{+\infty} E[x(t+\tau)\cdot x(t-a)]h_2(a)da \\
&= \int_{-\infty}^{+\infty} R_{xx}(\tau+a)\cdot h_2(a)da \\
&= R_{xx}(\tau) * h_2(-\tau)
\end{aligned}$$

Then we have:

$$R_{y_1y_2}(\tau) = R_{xx}(\tau) * h_2(-\tau) * h_1(\tau)$$

$$S_{y_1y_2}(w) = S_{xx}(w)\cdot H_2(-w)\cdot H_1(w)$$

$$\mu_{y_1} = \mu_x \int_{-\infty}^{+\infty} h_1(t)dt = 0 \quad \& \quad \mu_{y_2} = \mu_x \int_{-\infty}^{+\infty} h_2(t)dt = 0$$

$$C_{y_1y_2}(\tau) = R_{xx}(\tau) * h_2(-\tau) * h_1(\tau)$$

5. Consider an LTI system with system function:

$$H(s) = \frac{1}{s^2+4s+13}$$

The input to this system is a WSS process $X(t)$ with $\mathbb{E}\left[X^2(t)\right] = 10$. Find $S_X(\omega)$ such that the average power of output is maximum.
Solution:

For the system we know:

$$\begin{aligned}
H(s) &= \frac{1}{s^2+4s+13} \\
\Rightarrow H(\omega) &= \frac{1}{(jw)^2+4jw+13} \\
&= \frac{1}{(13-\omega^2)+j(4\omega)} \\
\Rightarrow |H(\omega)|^2 &= \frac{1}{(13-\omega^2)^2+(4\omega)^2} \\
&= \frac{1}{\omega^4+(16-26)\omega^2+169} \\
&= \frac{1}{(\omega^2-5)^2+144} \\
\Rightarrow \max|H(\omega)|^2 &= \frac{1}{144} \text{ at } \omega = \pm\sqrt{5}
\end{aligned}$$

We need S_{XX} to be concentrated on $\omega = \pm\sqrt{5}$. Therefore:

$$S_{XX}(\omega) = A\delta(\omega-\sqrt{5}) + B\delta(\omega+\sqrt{5})$$

We need to know $\int_{-\infty}^{\infty} S_{XX}(\omega)d\omega$. We can compute this from $\mathbb{E}\left[X^2(t)\right]$

$$\begin{aligned}
R_{XX}(0) &= \mathbb{E}\left[X^2(t)\right] = 10 \\
\Rightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)d\omega &= 10 \\
\Rightarrow \int_{-\infty}^{\infty} S_{XX}(\omega)d\omega &= 20\pi
\end{aligned}$$

And finally:

$$\left.\begin{array}{c} S_{XX}(\omega) = A\delta(\omega-\sqrt{5}) + B\delta(\omega+\sqrt{5}) \\ \int_{-\infty}^{\infty} S_{XX}(\omega)d\omega = 20\pi \end{array}\right\} \Rightarrow A+B = 20\pi$$

Any solution in form of $S_{XX}(\omega) = A\delta(\omega-\sqrt{5}) + B\delta(\omega+\sqrt{5})$ such that $A+B = 20\pi$ is correct.

6. Consider a WSS process $y(t)$ satisfying the equation

$$\frac{dy(t)}{dt} + 2y(t) = x(t)$$

where $x(t)$ is a zero-mean WSS process with covariance function:

$$C_{XX}(\tau) = \delta(\tau) + 4e^{-|\tau|}$$

Also assume that transformation from $x(t)$ to $y(t)$ is LTI. Determine $R_{XY}(\tau), S_{XY}(\omega)$.

Solution:

$$\left.\begin{array}{r} C_{xx}(\tau)=R_{xx}(\tau)-\mu_x^2 \\ \mu_x^2=0 \end{array}\right\} \Rightarrow R_{xx}(\tau)=C_{xx}(\tau)$$

$$\frac{dy}{dt}+2y(t)=x(t) \overset{F \cdot T}{\Longrightarrow} jwY(w)+2Y(w)=X(w)$$

$$\Rightarrow \frac{Y(\omega)}{X(\omega)}=\frac{1}{j\omega+2} \Rightarrow H(\omega)=\frac{1}{j\omega+2}$$

$$R_{xx}(\tau)=\delta(\tau)+4e^{-|\tau|} \xrightarrow{\text{F.T}} S_{xx}(\tau)=1+\frac{2\times 4}{\omega^2+1}$$

$$=1+\frac{8}{w^2+1}$$

$$H^*(w)=\left(\frac{1}{jw+2}\right)^*=\left(\frac{2-jw}{4+\omega^2}\right)^*=\frac{2+jw}{4+w^2}=\frac{1}{2-jw}$$

$$S_{XY}(\omega)=S_{XX}(\omega)H^*(w)=\left(1+\frac{8}{w^2+1}\right)\left(\frac{1}{2-j\omega}\right)$$

$$=\frac{w^2+9}{(w^2+1)(2-jw)}$$

$$R_{XY}(\tau)=F^{-1}\{S_{XY}(w)\}=L^{-1}\{S_{XY}(s)\}$$

$$S_{XY}(s)=\frac{9-s^2}{(1-s^2)(2-s)}=\frac{9-s^2}{(s^2-1)(s-2)}$$

$$=\frac{-4}{s-1}+\frac{\frac{4}{3}}{s+1}+\frac{\frac{5}{3}}{s-2}$$

$$\Rightarrow R_{xy}(\tau)=4e^t u(-t)+\frac{4}{3}e^{-t}u(t)-\frac{5}{3}e^{2t}u(-t)$$

7. The process $\mathbf{x}(t)$ is WSS with $\mathbb{E}[\mathbf{x}(t)]=5$ and $R_{xx}(\tau)=25+4e^{-2|\tau|}$. If $\mathbf{y}(t)=2\mathbf{x}(t)+3\mathbf{x}'(t)$, find $S_y(\omega)$

Solution:

$$Y(\omega)=(2+3j\omega)X(\omega) \Rightarrow H(\omega)=(2+3j\omega)$$

$$S_y(\omega)=S_x(\omega)|H(\omega)|^2=\left(50\pi\delta(\omega)+\frac{16}{4+\omega^2}\right)|2+3j\omega|^2.$$

8. We have an LTI system. The output of system is defined as:

$$y[n]=x[n]+x[n-2]-\frac{1}{2}x[n-3]+\frac{1}{2}y[n-1]$$

(a) Find the impulse response of the system in the time and frequency domains.

(b) Plot the impulse response in time domain from n=0 to n=5.

(c) If input is a zero-mean WSS process with autocorrelation defined as:

$$R_{XX}[m,n] = R_{XX}[l] = \delta[l] + \delta[|l-1|]$$

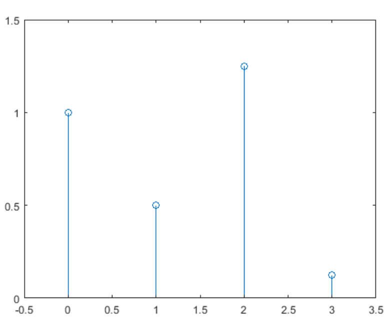
Find the R_{XY}.

Solution:

(a)

$$\begin{aligned}
y[n] &= x[n] + x[n-2] - \frac{1}{2}x[n-3] + \frac{1}{2}y[n-1] \\
\xrightarrow{F.T} Y(\omega) &= X(\omega) + e^{-2j\omega}X(\omega) - \frac{1}{2}e^{-3j\omega}X(\omega) + \frac{1}{2}e^{-j\omega}Y(\omega) \\
&= (1 + e^{-2j\omega} - \frac{1}{2}e^{-3j\omega})X(\omega) + \frac{1}{2}e^{-j\omega}Y(\omega) \\
\rightarrow H(\omega) &= \frac{Y(\omega)}{X(\omega)} = \frac{1 + e^{-2j\omega} - \frac{1}{2}e^{-3j\omega}}{1 - \frac{1}{2}e^{-j\omega}} \\
&= \frac{1}{1 - \frac{1}{2}e^{-j\omega}} + e^{-2j\omega} \\
\xrightarrow{I.F.T} h[n] &= 2^{-n}u[n] + \delta[n-2]
\end{aligned}$$

(b)



(c)

$$\begin{aligned}
R_{XY}[l] &= R_{XX}[l] * h[n] \\
&= 2^{-n}u[n] + \delta[n-2] + (2^{-n+1}u[n-1] + \delta[n-3]) + (2^{-n-1}u[n+1] + \delta[n-1]) \\
&= 2^{-n}u[n] + \delta[n-2] + (2^{-n+1}u[n] - 2^{-n+1}\delta[n] + \delta[n-3]) \\
&\quad + (2^{-n-1}u[n] + 2^{-n-1}\delta[n+1] + \delta[n-1]) \\
&= (1 + 2 + \frac{1}{2})2^{-n}u[n] + \delta[n-3] + \delta[n-2] + \delta[n-1] - 2\delta[n] + \frac{1}{4}\delta[n+1]
\end{aligned}$$