In the name of GOD.
Sharif University of Technology
Stochastic Processes CE 695 Fall 2022 H.R. Rabiee

Homework 2 Solutions (Stationary Stochastic Processes)

1. We define $X_0 = Y_0$ and $X_n = \lambda X_{n-1} + Y_n$ for $n \geq 1$. Suppose $Y_0, Y_1, ...,$ are uncorrelated random variables with $E[Y_n] = 0$ and $var(Y_n) = \begin{cases} \sigma^2/(1-\lambda^2), & n = 0 \\ \sigma^2, & n \geq 1 \end{cases}$ $where\ \lambda^2 < 1.$
Find $Cov(X_n, X_{n+m})$. Is this process WSS? why?

   Solution:
   The process $X_n, n \geq 0$ is called a first-order autoregressive process. It says that the state at time n (that is, X_n) is a constant multiple of the state at time n - 1 plus a random error term Y_n

   $X_N = \lambda(\lambda X_{n-2} + Y_{n-1}) + Y_n$
   $= \lambda^2 X_{n-2} + \lambda Y_{n-1} + Y_n$
   .
   .
   .
   $= \sum_{i=0}^{n} \lambda^{n-i} Y_i$

   $Cov(X_n, X_{n+m}) = Cov\Big(\sum_{i=0}^{n} \lambda^{n-i} Y_i, \sum_{i=0}^{n+m} \lambda^{n+m-i} Y_i\Big)$
   $= \sum_{i=0}^{n} \lambda^{n-i} \lambda^{n+m-i} Cov(Y_i, Y_i) = \dfrac{\sigma^2 \lambda^m}{1-\lambda^2}$

   where the preceding uses the fact that Y_i and Y_j are uncorrelated when i $\neq$ j. As $E[X_n] = 0$, we see that $X_n, n \geq 0$ is weakly stationary (the definition for a discrete time process is the obvious analog of that given for continuous time processes).

2. Prove that $Var(X(t+s) - X(t)) = 2R_X(0) - 2R_X(s)$. let X(t) be a WSS.

   Solution:
   $Var(X(t+s) - X(t))$
   $= Cov(X(t+s) - X(t), X(t+s) - X(t))$
   $= R(0) - R(s) - R(s) + R(0)$
   $= 2R(0) - 2R(s)$.

3. let X_i $(i \in Z)$ be a process in which the X_i's are i.i.d. supposing CDF $F_{X_i}(x) = F(x)$, prove that this process is SSS.

Solution:
Intuitively, since X(n)'s are i.i.d., we expect that as time evolves the probabilistic behavior of the process does not change. Therefore, this must be a stationary process. To show this rigorously, we can argue as follows. For all real numbers$x_1, x_2, ..., x_r$ and all distinct integers $n_1, n_2, .., n_r$, we have

$F_{X(n_1)X(n_2)...X(n_r)}(x_1, x_2, ..., x_r)$
$= F_{X(n_1)}(x_1)F_{X(n_2)}(x_2)...F_{X(n_r)}(x_r)$ (since the $X(n_i)$'s are independent)
$= F(x_1)F(x_2)...F(x_r)$ (since $F_{X(t_i)}(x) = F(x)$).

We also have
$F_{X(n_1+D)X(n_2+D)...X(n_r+D)}(x_1, x_2, ..., x_r)$
$= F_{X_{(n_1+D)}}(x_1)F_{X(n_2+D)}(x_2)...F_{X(n_r+D)}(x_r)$ (since the $X(n_i + D)$'s are independent)
$= F(x_1)F(x_2)...(x_n)(since F_{X(n_i+D)}(x) = F(x))$.

4. Consider the process $X(t) = Y(t + T)$ in which $T \sim U(0, T_0)$ and Y(t) is a periodic function with period T_0. Is this process SSS? Give reasons for your answer.

   Solution:
   It is SSS. Since this process is periodic the statics of the joint distribution remains the same and so the process is SSS.

5. Consider the process $X(t) = Y + Zt$ in wich Y and Z are normal N(1,1) and independent random variables. For this process find the correlation function and covariance function.

   Solution:
   $R_{X(t_1,t_2)} = E[X(t_1)X(t_2)]$
   $= E[(Y + Zt_1)(Y + Zt_2)]$
   $= E[Y^2] + E[YZ](t_1 + t_2) + E[Z^2]t_1t_2$
   $= 2 + E[Y]E[Z](t_1 + t_2) + 2t_1t_2$(since Y and Z are independent)
   $= 2 + t_1 + t_2 + 2t_1t_2$, for all $t_1, t_2 \in [0, \infty)$.

   And for the covariance we have:
   $C_{X(t_1,t_2)} = R_{X(t_1,t_2)}E[X(t_1)]E[X(t_2)]$
   $= 2 + t_1 + t_2 + 2t_1t_2(1 + t_1)(1 + t_2)$
   $= 1 + t_1t_2$, for all $t_1, t_2 \in [0, \infty)$.

6. Define X(t) = A cos(wt) + B sin(wt), where A and B are independent unit normal random variables and w is constant . Show that X(t) is a WSS.

   Solution:
   $Cov(X(t), X(t + s))$
   $= Cov(Acos(wt) + Bsin(wt), Acos(w(t + s)) + Bsin(w(t + s)))$

$= cos(wt)cos(w(t+s)) + sin(wt)sin(w(t+s))$
$= cos(w(t+s)wt)$
$= cos(ws)$.
and also since the mean is cte the process is WSS.

7. Suppose $X_1, X_2, ..$are i.i.d with $E[X_i] = 0$ and $var(X_i) = 4$. For the process $Y(n) = X_1 + X_2 + \ldots + X_n (n \in N)$ find the mean and covariance.

   Solution:
   $E[Y(n)] = E[X_1 + X_2 + \ldots + X_n] = E[X_1] + E[X_2] + ... + E[X_n] = 0$

   let $m \leq n$, then:
   $R_{Y(m,n)} = E[Y(m)Y(n)]$
   $= E[Y(m)(Y(m) + X_{m+1} + X_{m+2} + ... + X_n)]$
   $= E[Y(m)^2] + E[Y(m)]E[X_{m+1} + X_{m+2} + ...X_n]$
   $= E[Y(m)^2] + 0 = Var(Y(m))$
   $= Var(X_1) + Var(X_2) + ... + Var(X_m)$
   $= 4m$.

   Similarly, for $m \geq n$, we have $R_{Y(m,n)} = E[Y(m)Y(n)] = 4n$
   We conclude $R_{Y(m,n)} = 4min(m, n)$.