In the name of GOD.
Sharif University of Technology
Stochastic Processes CE 695 Fall 2022 H.R. Rabiee

Homework 1 (Review of Probability)

1. There is a coin that comes up heads with probability p. The only thing we know about p is that it comes from the $Uniform(0, 1)$.
   (a) Calculate the probability that in 5 coin tosses all the tosses are heads.
   (b) Assuming that the first 4 tosses are heads, calculate the probability that the next toss is head.

2. Let X_1 and X_2 be independent exponential random variables with rates λ_1 and λ_2, respectively.
   (a) Show that the conditional density function of X_1, given $X_1+X_2 = 2$, is given by,

$$f_{X_1|X_1+X_2}(x|2) = \frac{(\lambda_1-\lambda_2)e^{-(\lambda_1-\lambda_2)x}}{1-e^{-2(\lambda_1-\lambda_2)}}, \quad 0 < x < 2$$

   (b) Find $E[X_1|X_1 + X_2 = 2]$

3. Toss a coin until it lands twice in a head or twice in a tail. Based on Markov's inequality:
   (a) Find an upper bound for the probability that the number of tosses is at most less than 6.
   (b) Find a lower bound for the probability that the number of tosses is less than 9.

4. Prove the following equation by applying the central limit theorem to a sequence of independent random variables with Poisson distribution with parameter $\lambda = 1$.

$$lim_{n\to\infty} e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!} = \frac{1}{2}$$

5. Let X and Y be jointly distributed random variables and define the new random variable $Z = XY$. Show that $Var(Z|Y) = Y^2 Var(X|Y)$.

6. We know that random variable X with $Uniform(-1, 1)$ distribution and random variable $Y = X^2$ is dependent on each other.
   (a) Calculate the covariance between X and Y.

(b) Is there a conflict between the value in the previous section and the X and Y dependency? discuss.

7. We have n beads with numbers 1 to n in a bag. At each step, a person randomly takes a bead out of the bag (with equal probability for different beads), looks at a number on it, and throws it back into the bag. What is the expected value of steps to see all the numbers?

8. (a) Prove the weak law of large numbers using Chebyshev's theorem.

   (b) Briefly explain why the strong law of large numbers(SLLN) implies the weak law of large numbers(WLLN).