In the name of GOD.
Sharif University of Technology
Stochastic Processes CE 695 Fall 2022 H.R. Rabiee

Homework 4 (Point Process and Gaussian Process)

1. Suppose $X(t)$ is a Gaussian process with zero mean and kernel $k(t_1, t_2) = 2\exp - \frac{|t_1 - t_2|}{\alpha}$. Let $Y(t)$ be the result of the application of filter $H(\omega) = 1 + 0.5e^{-2j\omega}$ on $x(t)$.
   (a) Compute mean and auto-covariance of $Y(t)$
   (b) Prove that $Y(t)$ is a Gaussian process.

2. Assume that K_1 and K_2 kernels be represented as $K_i(x, y) = \phi_i(x)^T \phi_i(y)$ for i = 1, 2 where ϕ_i is a mapping that maps input onto a higher dimensional space. Note that ϕ_i can have infinite dimensions. We call such kernels as valid kernels.
   (a) Prove that $K_1 + K_2$ is also a valid kernel.
   (b) Prove that $K_1 \times K_2$ is also a valid kernel.
   (c) Prove that e^{K_1} is also valid.

3. Suppose that $N(t)$ is a Poisson process with parameter λ.
   (a) Calculate auto-covariance function. Is $N(t)$ W.S.S?
   (b) Calculate p.d.f of $(N(t_1), N(t_2))$.
   (c) For $t_0, t_1, t_2, ..., t_n$ find the p.d.f of $(N(t_0), ..., N(t_n)) = f(a_0, ..., a_n)$.
   (d) From the previous question, for $t_i = t + in$ find

   $$argmax_{a_1,...,a_{n-1}} f(a, a_1, ..., a_{n-1}, a + kn)$$

   (e) Find the probability that $N(4) < 3$ and $N(5) - N(3) > 3$.

4. Suppose $X(t)$ is a Gaussian process, with $X(t) = 0$ with probability 1. Suppose that $X_t + X_s \sim \mathcal{N}(0, \sqrt{|t - s|})$
   (a) Calculate the auto-covariance function.
   (b) Calculate the distribution of $(X(t_1), X(t_2), ..., X(t_n))$
   (c) Prove that such a process doesn't exists.

5. A machine needs frequent maintenance to stay on. The maintenance times occur as a Poisson process with rate μ. Once the machine receives no maintenance for a time interval of length h, it breaks down. It then needs to be repaired, which takes an Exponential(λ) time, after which it goes back on.

(a) After the machine is started, find the probability that the machine will break down before receiving its first maintenance.

(b) Find the expected time for the first breakdown.

(c) Find the proportion of time the machine is on.

6. A car wash has two stations, 1 and 2, with Exponential (λ_1) and Exponential (λ_2) service times. A car enters at station 1. Upon completing the service at station 1, the car then proceeds to station 2, provided station 2 is free; otherwise, the car has to wait at station 1, blocking the entrance of other cars. The car exits the wash after service at station 2 is completed. When you arrive at the wash there is a single car at station 1. Compute your expected time before you exit.

7. Let (N_t) a Poisson process with intensity λ. We define

$$G_t = t - T_{N_t} \quad , \quad D_t = T_{N_t+1} - t$$

(a) For a given t and $0 < x \leq t, y \geq 0$, show that $(G_t < x, D_t \leq y) = (N_{tx} < N_t < N_{t+y})$. Find $\mathbb{P}(G_t < x, D_t \leq y)$.

(b) For a given t and $y > 0$, show that $(G_t = t, D_t \leq y) = (N_t = 0, N_{t+y} > 0)$. Find $\mathbb{P}(G_t = t, D_t \leq y)$.

(c) For a given t and $y > 0$, calculate $(D_t \leq y)$ and find the distribution of D_t.

(d) Calculate the cumulative distribution function of G_t.

(e) Calculate $\mathbb{P}(\min(T_1, t) > x)$ for all $x \in \mathbb{R}$. Deduce that G_t has the same distribution as $\min(T_1, t)$.

(f) Show that G_t and D_t are independent.

(g) Calculate $\mathbb{E}[G_t]$. Deduce $\mathbb{E}[G_t + D_t]$. What do you think about this result?